Dimostrare che $\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$ $\quad \forall a,b \in \mathbb{R}^+$

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- partiamo da $\boxed{\sqrt{ab} \leq \frac{a+b}{2}}$ (*) : $\quad \sqrt{ab} \leq \frac{a+b}{2} \iff 2\sqrt{ab} \leq a+b$

  $\iff (\sqrt{a}-\sqrt{b})^2 \geq 0$ VERO $\forall a,b \in \mathbb{R}^+$

- $\frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2ab}{a+b} \leq \sqrt{ab} \iff a+b \geq \frac{2ab}{\sqrt{ab}} = 2\sqrt{ab}$ VERO $\forall a,b \in \mathbb{R}^+$ per (*)

- $\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \iff \frac{(a+b)^2}{4} \leq \frac{a^2+b^2}{2} \iff a^2+b^2-2ab = (a-b)^2 \geq 0$ VERO $\forall a,b \in \mathbb{R}^+$

N.B. inoltre, vale anche $|ab| \leq \frac{a^2+b^2}{2}$

Risolvere in $\mathbb{C}$:

1) $\dfrac{1}{|z|\bar{z}} = e^{i\frac{\pi}{4}}$

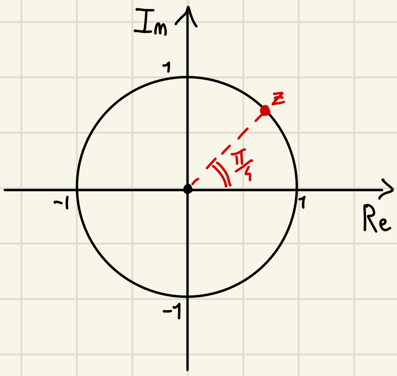
2) $\bar{z}\left(Im(z) - Re(z)\right) = z$

3) $z^3 = 1$

4) $2z^2 + \bar{z} = -1$

1) • prendiamo il modulo: $\left|e^{i\frac{\pi}{4}}\right| = \left|\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right| = \left|\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right| = 1$

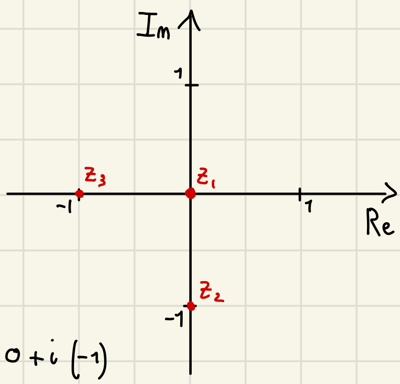
$\Rightarrow 1 = \left|\dfrac{1}{|z|\bar{z}}\right| \overset{|z|=|\bar{z}|}{=} \dfrac{1}{|z|^2} \quad \Rightarrow \quad |z| = 1$



• sostituiamo: $\dfrac{1}{\bar{z}} = e^{i\frac{\pi}{4}}$

• moltiplichiamo e dividiamo per z: $e^{i\frac{\pi}{4}} = \dfrac{z}{\bar{z}\cdot z} = \dfrac{z}{|z|} = z$

2) • $z = a + ib \quad a, b \in \mathbb{R} \quad \Rightarrow \begin{cases} \bar{z} = a - ib \\ Re(z) = a \\ Im(z) = b \end{cases}$



$\Rightarrow (a - ib)(b - a) = a + ib \quad \Rightarrow \begin{cases} a(b-a) = a \\ -b(b-a) = b \end{cases}$

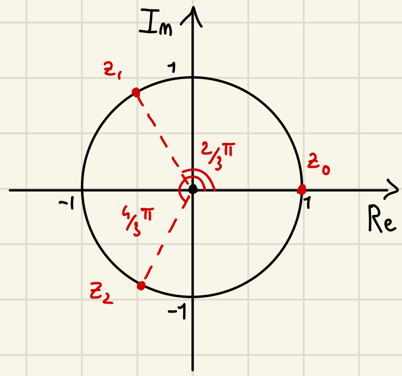
• se $a = 0$, otteniamo $b(1+b) = 0 \Rightarrow z_1 = 0 + i\cdot 0 \qquad z_2 = 0 + i(-1)$

se $a \neq 0$, " $\begin{cases} a = b - 1 \\ -b = b \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 0 \end{cases} \Rightarrow z_3 = -1 + i(0)$

3) • Th, le radici n-esime di $w = \rho e^{i\theta}$ sono $w_k = \sqrt[n]{\rho}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+i\sin\left(\frac{\theta+2\pi k}{n}\right)\right]$ $\quad k=0,1,\dots,n-1$

• $\Rightarrow z^3 = 1\cdot e^{i\cdot 0} \quad \Rightarrow z_k = \overset{\sqrt[3]{|1|}}{1}\left[\cos\left(\frac{2\pi k}{3}\right)+i\sin\left(\frac{2\pi k}{3}\right)\right] \quad k=0,1,2$

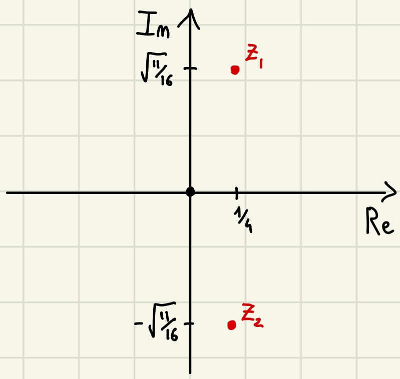
$\Rightarrow z_1 = 1\cdot e^{i0} \qquad z_2 = 1\cdot e^{i\frac{2}{3}\pi} \qquad z_3 = 1\cdot e^{i\frac{4}{3}\pi}$



4) • $z = a+ib \quad a,b\in\mathbb{R} \quad \Rightarrow \begin{cases} z^2 = (a+ib)(a+ib) = (a^2-b^2)+i(2ab) \\ \bar{z} = a-ib \end{cases}$

$\Rightarrow 2(a^2-b^2)+4iab+(a-ib) = -1 \quad \Rightarrow \begin{cases} 2a^2-2b^2+a = -1 \\ 4ab-b = b(4a-1) = 0 \end{cases}$

• $\begin{cases} \text{se } b=0, & \begin{cases} 2a^2+a+1=0 \\ b=0 \end{cases} \rightsquigarrow \Delta = 1-8<0 \Rightarrow \nexists \text{ soluzione} \\ \text{se } b\neq 0, & \begin{cases} \frac{1}{8}-2b^2+\frac{1}{4} = -1 \\ a = \frac{1}{4} \end{cases} \rightsquigarrow \begin{matrix} b = \pm\sqrt{\frac{11}{16}} \\ a = \frac{1}{4} \end{matrix} \end{cases}$



Risolvere in $\mathbb{C}$:

1) $z^6 + i\,\bar{z}^3 = 0$

2) $z^2 - (3+i)z + 2 = 0$

3) $\begin{cases} \bar{z}^2 - 2w^2 + 2 = 0 \\ \bar{w}^2 - z = 0 \end{cases}$

4) $z^2 |z| = i$

1) • $|z^6| = |-i\bar{z}^3| = |z^3| \Rightarrow |z|^3(|z|^3 - 1) = 0 \Rightarrow |z| = 0 \vee |z| = 1$

• $\begin{cases} \text{se } |z| = 0, \quad z = 0 \\ \text{se } |z| = 1, \quad z^3(z^6) = z^3(-i\bar{z}^3) = -i\,|z|^3 = -i \quad \Rightarrow z^9 = -i \\ \quad \Rightarrow \text{Th}, \quad z_k = \sqrt[9]{1}\, e^{i\left(\frac{3}{18}\pi + \frac{2}{9}k\pi\right)} \quad k = 0, 1, \dots, 8 \end{cases}$

N.B. $-i = 1 \cdot e^{i\frac{3}{2}\pi}$

2) • usiamo la formula del 2° grado: $z_{1,2} = \dfrac{(3+i) \pm \sqrt{(3+i)^2 - 8}}{2} = \dfrac{(3+i) \pm \sqrt{6i}}{2}$

• $\sqrt{i} = \left(e^{i\frac{\pi}{2}}\right)^{1/2} = e^{i\left(\frac{\pi}{4} + k\pi\right)} \quad k = 0, 1 \quad \Rightarrow \sqrt{i} = \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

• sostituiamo: $z_{1,2} = \dfrac{(3+i) \pm \sqrt{6}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)}{2} = \dfrac{3 \pm \sqrt{3}}{2} + i\,\dfrac{1 \pm \sqrt{3}}{2}$

3) • dalla 2° eq., $\left(\overline{w}^2 = z \iff w^2 = \overline{z}\right)$ e quindi $\begin{cases} \overline{z}^2 - 2\overline{z} + 2 = 0 \\ w^2 = \overline{z} \end{cases}$ N.B. $w = \sqrt{\overline{z}}$ è abuso di notazione

• formula: $\overline{z}_{1,2} = 1 \pm \sqrt{1-2} = 1 \pm i \Rightarrow z_{1,2} = 1 \mp i$ N.B. $|z_{1,2}| = \sqrt{2}$ e $\begin{cases} z_1 = \sqrt{2}\, e^{-i\frac{\pi}{4}} \\ z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}} \end{cases}$

• $\Rightarrow \begin{cases} z = 1-i \\ w = \sqrt{\sqrt{2}}\; e^{i\left(-\frac{\pi}{8}+K\pi\right)} \end{cases} K=0,1 \quad \vee \quad \begin{cases} z = 1+i \\ w = \sqrt[4]{2}\; e^{i\left(\frac{\pi}{8}+K\pi\right)} \end{cases} K=0,1$

4) • prendiamo il modulo: $\left|z^2 |z|\right| = |z|^3 = |i| = 1 \Rightarrow |z| = 1$

• abbiamo $z^2 = i \quad \left(= e^{i\frac{\pi}{2}}\right) \Rightarrow \begin{cases} z_0 = 1 \cdot \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right] = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\ z_1 = 1 \cdot \left[\cos\left(\frac{5}{4}\pi\right) + i \sin\left(\frac{5}{4}\pi\right)\right] = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \end{cases}$

• alternativamente, poniamo $z = a+ib$ $a,b \in \mathbb{R}$ e risolviamo $(a+ib)^2\sqrt{a^2+b^2} = i$

$$\Rightarrow \begin{cases} (a^2-b^2)\sqrt{a^2+b^2} = 0 \\ 2ab\sqrt{a^2+b^2} = 1 \end{cases}$$

dalla 1° eq., $\begin{cases} (a=b=0) \longrightarrow \begin{cases} a=b=0 \\ 0 \cdot 0 = 1 \end{cases} \text{ IMPOSSIBILE} \\ \vee \\ (a^2 = b^2) \end{cases}$

$(a^2=b^2) \longrightarrow a = b \longrightarrow \begin{cases} a = b > 0 \\ 2\sqrt{2}\, a^3 = 1 \end{cases} \Rightarrow \begin{cases} a = b \\ a = \frac{1}{\sqrt[3]{2}\,\sqrt[6]{2}} = \frac{\sqrt{2}}{2} \end{cases}$

$a = b \longrightarrow \begin{cases} a = b < 0 \\ -2\sqrt{2}\, a^3 = 1 \end{cases} \Rightarrow \begin{cases} a = b \\ a = -\frac{\sqrt{2}}{2} \end{cases}$

$(a^2=b^2) \longrightarrow \begin{cases} a = -b \\ -2\sqrt{2}\, a^2\sqrt{a^2} = 1 \end{cases} \Rightarrow \begin{cases} a = -b \\ a^2|a| = -\frac{1}{2\sqrt{2}} \end{cases}$ IMPOSSIBILE

Dati $f, g, h : \mathbb{C} \longrightarrow \mathbb{C}$ definite da $f(z) = z + z_0$ ($z_0 \in \mathbb{C}$ fissato), $g(z) = iz$ e $h(z) = z^2$

$A = \{ z \in \mathbb{C} : \ Re(z), Im(z) \in [0,1] \}$

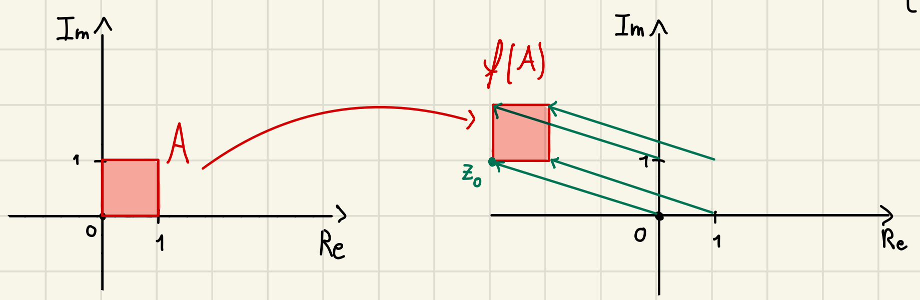
$B = \{ \ " \ : \ |z| \leq 2, \ arg(z) \in (\frac{3}{4}\pi, \pi) \}$

$C = \{ \ " \ : \ 1 \leq |z| \leq 2 \}$

rappresentare $f(A)$, $g(B)$, $h(B)$, $h(C)$ descrivendone l'azione su A, B, C

$f(A)$ $\quad z_0 = a_0 + i b_0$ e $A = \{ a + ib \in \mathbb{C} : \ a, b \in [0,1] \} \implies f(A) = \{ f(a+ib) : \ a, b \in [0,1] \}$

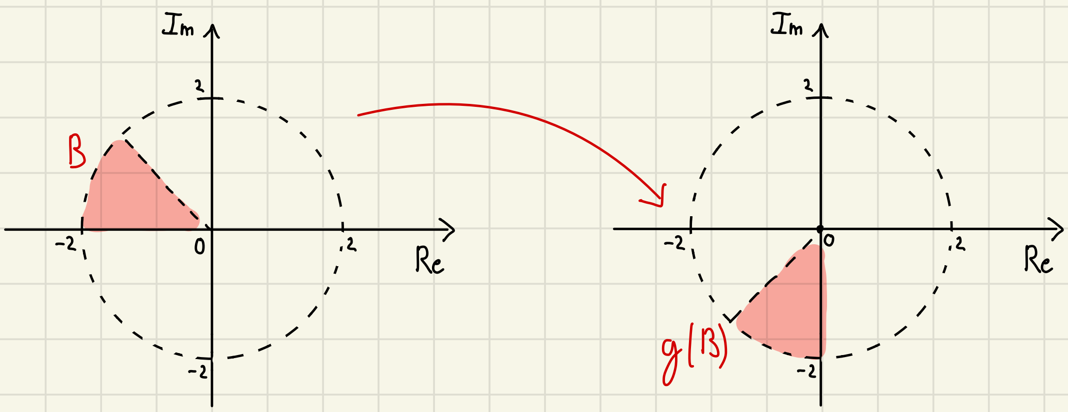
$= \{ (a + a_0) + i(b + b_0) : \ a, b \in [0,1] \}$



f è una TRASLAZIONE

$g(B)$ $\quad g(z) = iz \implies z = |z| [\cos(arg(z)) + i \sin(arg(z))] \longmapsto g(z) = |z| [\cos(arg(z) + \frac{\pi}{2}) + i \sin(arg(z) + \frac{\pi}{2})]$

$\implies g(B) = \{ z \in \mathbb{C} : \ |z| \leq 2, \ arg(z) \in (\frac{5}{4}\pi, \frac{3}{2}\pi) \} \implies g$ è ROTAZIONE di $\frac{\pi}{2}$

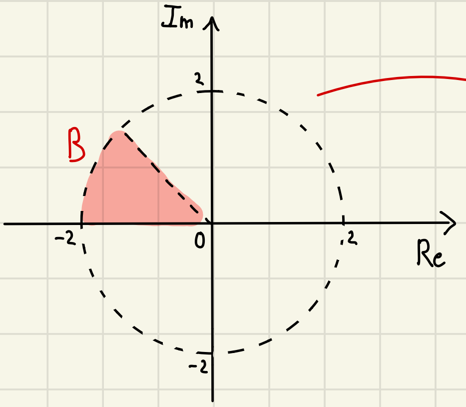


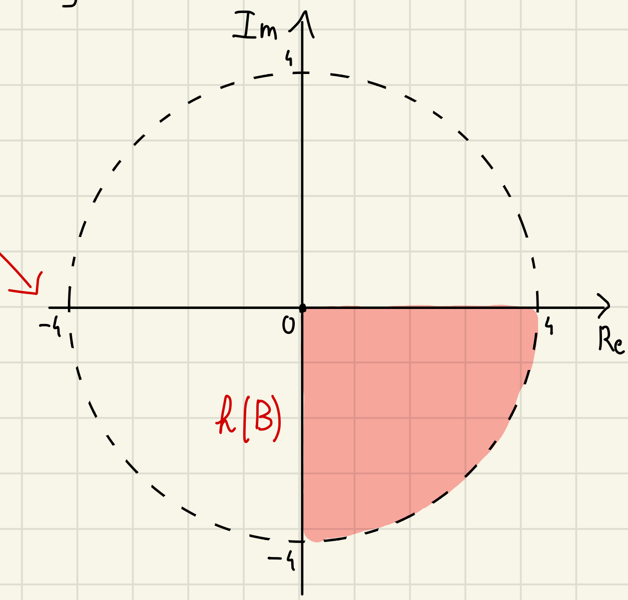
$h(B)$ similmente, $z = |z|\left[\cos(\arg(z)) + i\sin(\arg(z))\right] \xmapsto{h} z^2 = |z|^2\left[\cos(2\cdot\arg(z)) + i\sin(2\cdot\arg(z))\right]$

$$\Rightarrow \begin{cases} |h(z)| \leq 4 & (\text{perché } |z| \leq 2 \text{ per } z \in B) \\ \frac{3}{2}\pi < 2\cdot\arg(z) < 2\pi & (\text{" } \arg(z) \in \left]\frac{3}{4}\pi, \pi\right[\text{ per } z \in B) \end{cases}$$

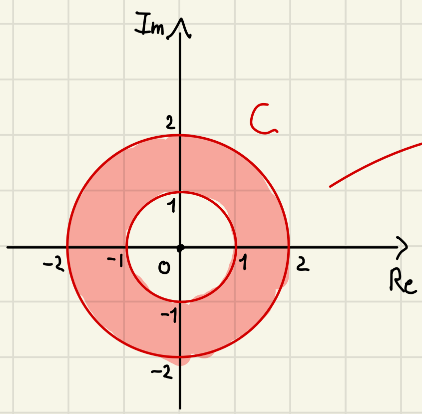
$$\Rightarrow h(B) = \left\{ z \in \mathbb{C} : \ |z| \leq 4 \ , \ \arg(z) \in \left]\tfrac{3}{2}\pi, 2\pi\right[\right\}$$

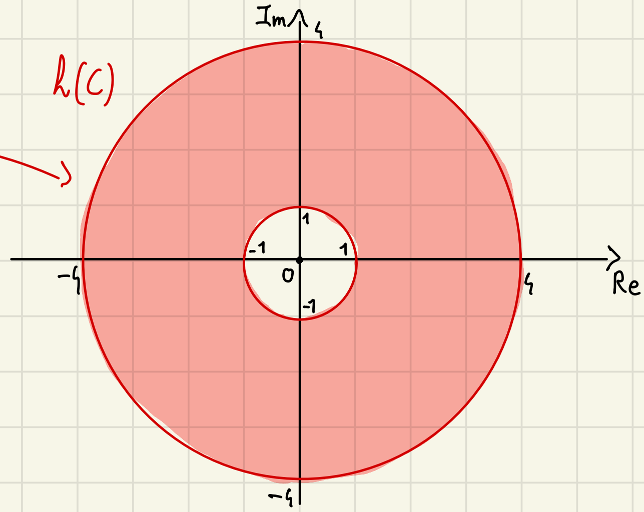
h è ROTAZIONE + DILATAZIONE





$h(C)$ $C = \{ z \in \mathbb{C} : \ 1 \leq |z| \leq 2 \}$ $\Rightarrow$ $h(C) = \{ z \in \mathbb{C} : \ 1 \leq |z| \leq 4 \}$





Rappresentare in $\mathbb{C}$ i seguenti insiemi:

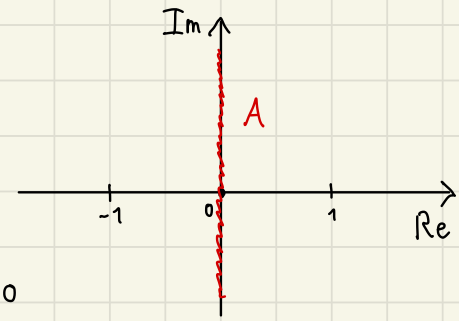
1) $A = \left\{ z \in \mathbb{C} : \dfrac{|z-1|}{|z+1|} = 1 \right\}$

2) $B = \left\{ z \in \mathbb{C} : |z-i-2| < |z+2| \, , \, |z| \geq 1 \right\}$

1) • $|z-1| = |z+1| \iff |a-1+ib| = |a+1+ib|$

$\iff (a-1)^2 + \cancel{b^2} = (a+1)^2 + \cancel{b^2}$

$\iff \cancel{a^2} \cancel{+1} - 2a = \cancel{a^2} \cancel{+1} + 2a \iff a = 0$



2) • dalla condizione $|z| \geq 1$, i punti di B sono esterni al cerchio centrato nell'origine di raggio 1 (a parte quelli del bordo)

• studiamo l'altra condizione ponendo $z = a + ib \quad a, b \in \mathbb{R}$

$$|a-2+i(b-1)| < |a+2+ib| \iff (a-2)^2 + (b-1)^2 < (a+2)^2 + b^2$$

$$\iff \cancel{a^2} \cancel{+4} - 4a + \cancel{b^2} + 1 - 2b < \cancel{a^2} \cancel{+4} + 4a + \cancel{b^2}$$

$$\iff \frac{1-2b}{8} < a$$