parameter

$\boxed{\Theta}$ — $\underline{x}$ ← measurement / statistic

$\underline{x} \to \boxed{\text{Estimator}} \to \hat{\Theta}$ ← estimate

$x_i = \theta + n_i$

$x_1 = \theta + n_1$

$x_2 = \theta + n_2$

$\vdots$

$x_N = \theta + n_N$

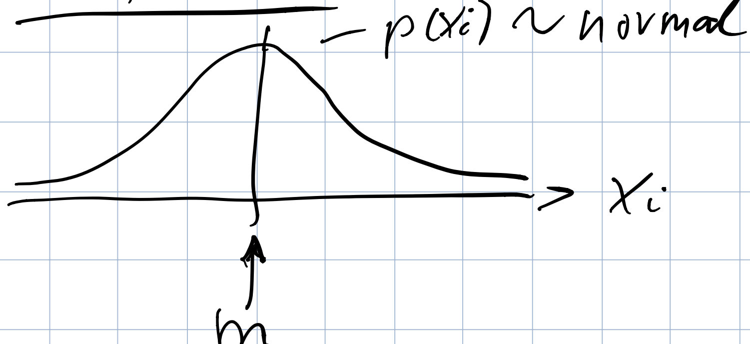
$\hat{\Theta} = f(\underline{x})$

$\frac{1}{n}\sum x_i \to \hat{\Theta}$

Example: sample mean $\hat{m}_x = \frac{1}{n}\sum_{i=1}^{n} x_i$ good estimate?

x_i ← random variable

Distribution



random quantity

$\hat{m}_x \to m$ $\quad Pr(|\hat{m}_x - m| < \varepsilon) \to 1$

by law of large numbers

$$\boxed{\text{Estimator } \hat{\Theta} = g(\underline{x})}$$

random variables

- Unbiased estimator $E(\hat{\Theta}) = \Theta$

$$\hat{m}_x = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Variance of $\hat{m}_x$

$$\hat{\sigma}_x^2 = \cancel{\frac{1}{n}} \sum (x_i - \hat{m}_x)^2$$

$$\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{m}_x)^2$$

Why is $\hat{\sigma}_x^2$ unbiased:

$$E((x_i - \hat{m}_x)^2) = E(x_i^2) - 2E[x_i \hat{m}_x] + E[\hat{m}_x^2]$$

$$\hat{m}_x = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$= E(x_i^2) - 2E\left(x_i \frac{1}{n}\sum_{j=1}^{n} x_j\right)$$

$$+ E\left(\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\right)$$

$$E(x_i^2) - \frac{1}{n}2\left[E[x_i^2] - \underset{E[x_i]E[x_j]}{E[x_i x_j]}\cdot n-1\right]$$

$$-\frac{2}{n}E(x_i^2) - \frac{2}{n}m^2(n-1)$$

$$E(x_i^2)\left(1-\frac{2}{n}\right) - 2\frac{n-1}{n}m^2$$

$$\underbrace{+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E(x_i x_j)}$$

$i \neq j$

$$+\frac{1}{n^2}\left[n\,E[x_i^2] + n(n-1)\,E[x_i x_j]\right]$$

$$\frac{1}{n}E(x_i^2) + \frac{n-1}{n}\cdot m^2$$

$$E(x_i^2)\left(1-\frac{1}{n}\right) - \left(1-\frac{1}{n}\right)m^2 = \left(1-\frac{1}{n}\right)\underbrace{(E(x_i^2)-m^2)}_{\sigma^2}$$

$$E\left((x_i-\hat{m}_x)^2\right) = \left(1-\frac{1}{n}\right)\cdot\sigma^2$$

$$\underbrace{\boxed{\hat{\sigma}_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\hat{m}_x)^2} \rightarrow \sigma^2}$$

unbiased estimator

"Goodness" of estimator

$$\hat{\theta} \to \theta$$

$E((\hat{\theta}-\theta)^2)$ — minimize "energy"

minimum-variance estimators

$$= E((\theta - E(\hat{\theta}) + E(\hat{\theta}) - \hat{\theta})^2)$$

just number

$$= E((\theta - E(\hat{\theta})) - (\hat{\theta} - E(\hat{\theta}))^2)$$

$$\underbrace{E((\theta-\hat{\theta})^2)}_{\text{min. var. estimator}} = \underbrace{(\theta - E(\hat{\theta}))^2}_{} + \operatorname{var}(\hat{\theta})$$

$E((\hat{\theta}-E(\hat{\theta}))^2)$

requires θ <u>itself</u>

$= 0$ avoids problem.

$E(\hat{\theta}) = \theta$ unbiased

Minimum-variance unbiased estimator

→ optimal estimator "good estimator"

Cramer Rao Lower Bound (CRLB)

- classical estimation

  θ fixed $\longrightarrow$ $\underline{X}(\theta)$
- "optimal estimator" MVUE
- $\underline{\theta}$ multiple parameters

- Coin toss which is biased : $Pr(H) = \theta \neq 0.5$

  $Pr(T) = 1 - \theta$

$\underline{X}$ measurments $\underbrace{X_1, X_2 \cdots\cdots X_n}_{\text{coin tosses}}$

$\{H, T\}$

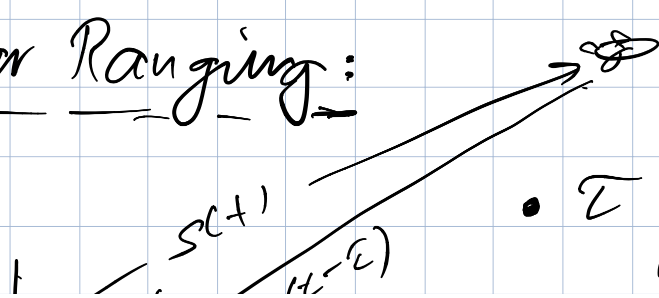
$\downarrow$

$\{1, 0\}$

Statistics $\longrightarrow \hat{\theta}$

$$Z = \underbrace{\frac{X_1 + X_2 + \cdots X_n}{n}}_{\text{estimator}} = \hat{\theta}$$

Radar Ranging:



- τ is the parameter

$\to$ distance

- $\boxed{\hat{\tau} = f(\underline{x})}$

$\hookrightarrow x_1 x_2 \cdots x_N = \underline{x}$