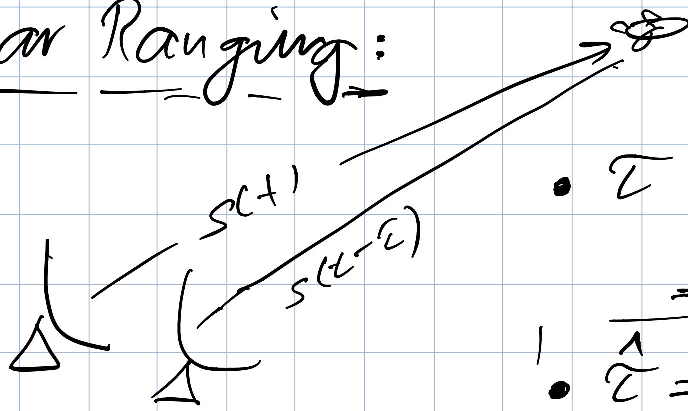
Radar Ranging:



$\hookrightarrow x_1 x_2 \cdots x_N = \underline{x}$

- τ is the parameter → distance
- $\boxed{\hat{\tau} = f(\underline{x})}$

Bound for $E((\hat{\theta}-\theta)^2) \geq$ lower bound

$\hat{\theta}$ = unbiased $\quad \hat{\theta}(\underline{x})$

$$E(\hat{\theta}-\theta) = \int_{\underline{x}} (\hat{\theta}(\underline{x})-\theta)\, f_{\underline{x}}(\underline{x};\theta) = 0$$

over $\hat{\theta}$

over $\underline{x}$

$E(\hat{\theta})=\theta$

$$\boxed{\theta} \longrightarrow \underline{x} \longrightarrow \boxed{g(\underline{x})} \longrightarrow \hat{\theta}$$

$\underline{x} \hookrightarrow$ random $\quad f_{\underline{x}}(\underline{x};\theta)$

pdf w. parameter

$$\frac{d}{d\theta}\int_{\underline{x}} (\hat{\theta}-\theta)\, f_x(\underline{x};\theta) = \int_{\underline{x}} \frac{d}{d\theta}\, \underline{(\hat{\theta}-\theta)}\, \underline{f_x(\underline{x};\theta)}$$

$$= -\underbrace{\int_{\underline{x}} f_{\underline{x}}(\underline{x};\theta)} + \int_{\underline{x}} (\hat{\theta}-\theta)\cdot \frac{d}{d\theta} f_{\underline{x}}(\underline{x};\theta)$$

$$1 = \int_{\underline{x}} (\hat{\theta}-\theta)\cdot \underbrace{\frac{d}{d\theta} f_{\underline{x}}(\underline{x};\theta)}$$

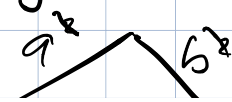
$$\frac{d \ln(f_{\underline{x}}(\underline{x};\theta))}{d\theta} f_{\underline{x}}(\underline{x};\theta)$$

$$1 = \int_{\underline{x}} (\hat{\theta}-\theta) \frac{d}{d\theta} \ln(f_{\underline{x}}(\underline{x};\theta)) f_{\underline{x}}(\underline{x};\theta)$$

$$\sqrt{f_{x}(\underline{x};\theta)} \cdot \sqrt{f_{\underline{x}}(\underline{x};\theta)}$$

$$1 = \int_{\underline{x}} \underbrace{(\hat{\theta}-\theta)\cdot \sqrt{f_{x}(\underline{x};\theta)}}_{g(x)} \underbrace{\frac{d}{d\theta} \ln(f_{\underline{x}}(\underline{x};\theta)) \sqrt{f_{x}(\underline{x};\theta)}}_{h(x)}$$

$\int (g(x)-h(x))^2 dx \longrightarrow$ triangle inequality



c^2

$c \leq a+b$

$$\left| \int_{\underline{x}} g^*(\underline{x})\, h(\underline{x}) \right|^2 \leq \int_{\underline{x}} g^2(\underline{x}) \int h^2(\underline{x})$$

Schwarz - Inequality

$$1 \leq \int_{\underline{x}} \underbrace{(\hat{\theta}-\theta)^2 f_{\underline{x}}(\underline{x};\theta)}_{} \int_{\underline{x}} \left(\frac{d \ln f_{\underline{x}}(\underline{x};\theta)}{d\theta}\right)^2 f_{\underline{x}}(\underline{x};\theta)$$

$E(\hat{\theta})$ PDF PDF

$$1 \leq var(\hat{\theta}) \quad \boxed{}$$

$$var(\hat{\theta}) \geq \left[\int_{\underline{x}} \left(\frac{d \ln f_{\underline{x}}(\underline{x};\theta)}{d\theta}\right)^2 f_{\underline{x}}(\underline{x};\theta) \right]^{-1}$$

CRLB:

$$E((\hat{\theta}-\theta)^2) \geq E\left[\left(\frac{d \ln f_{\underline{x}}(\underline{x};\theta)}{d\theta}\right)^2\right]^{-1} = \frac{1}{I(\theta)}$$

$$I(\theta) = E\left[\left(\frac{d \ln f_{\underline{x}}(\underline{x};\theta)}{\ldots}\right)^2\right]$$

$$\frac{d\theta}{} \Big)\Big]$$

Fisher Information

- $$I(\theta) = -\int_{\underline{x}} \frac{d^2 \ln(f_{\underline{x}}(\underline{x};\theta))}{d\theta^2} f_x(\underline{x};\theta)$$

- when do we have equality

  from Schwartz $g(x) = c \cdot h(x)$

$$\left[\frac{d \ln(f_{\underline{x}}(\underline{x};\theta)}{d\theta}\right]^2 = c \cdot (\hat{\theta}-\theta)^2$$

Not typically the case

$$\underbrace{\frac{d \ln(f_x(\underline{x};\theta))}{d\theta} = I(\theta)(\hat{\theta}-\theta)}_{\text{equality in the CRLB}}$$

optimal

Noisy measurements:

Gaussian

$x_i = A + n_i \Rightarrow x_i - A$

$\vdots$

$$f_{\underline{X}}(\underline{x};A) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-A)^2/2\sigma^2}$$

$x_1, x_2, \dots, x_n$

$$f_{\underline{X}}(\underline{x};A) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\sum_{i=1}^{n}(x_i-A)^2/2\sigma^2}$$

$$\Rightarrow \frac{d}{dA}\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-A)^2\right)$$

$$+\frac{2}{2\sigma^2}\sum_{i=1}^{n}(x_i-A)$$

$$\frac{d\ln(f_{\underline{X}}(\underline{x};A))}{dA} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-A)$$

$$= \frac{n}{\sigma^2}\Big(\underbrace{\frac{1}{n}\sum_{i=1}^{n}x_i}_{\sim} - A\Big)$$

$$= \frac{n}{\sigma^2}(\hat{A} - A)$$

equality condition

optimal estimator

$$\hat{A} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{A} = \frac{1}{n}\sum_{i=1} x_i$$

$$I(A) = \frac{n}{\sigma^2}$$

$$\operatorname{var}(\hat{A}) = \frac{1}{I(A)} = \frac{\sigma^2}{n} \quad (\checkmark)$$

<u>not typical</u> !!