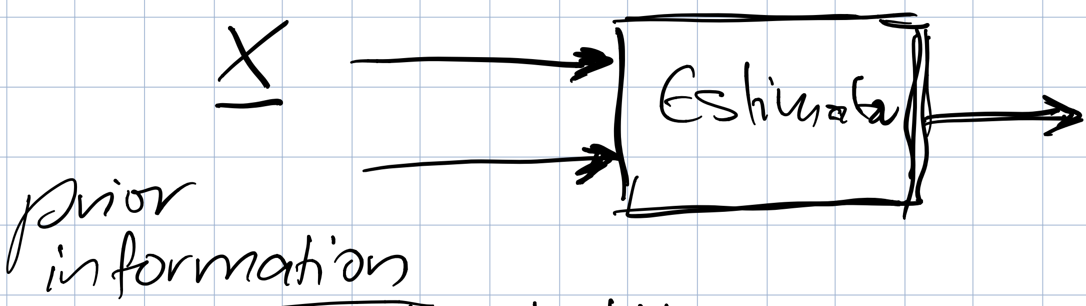
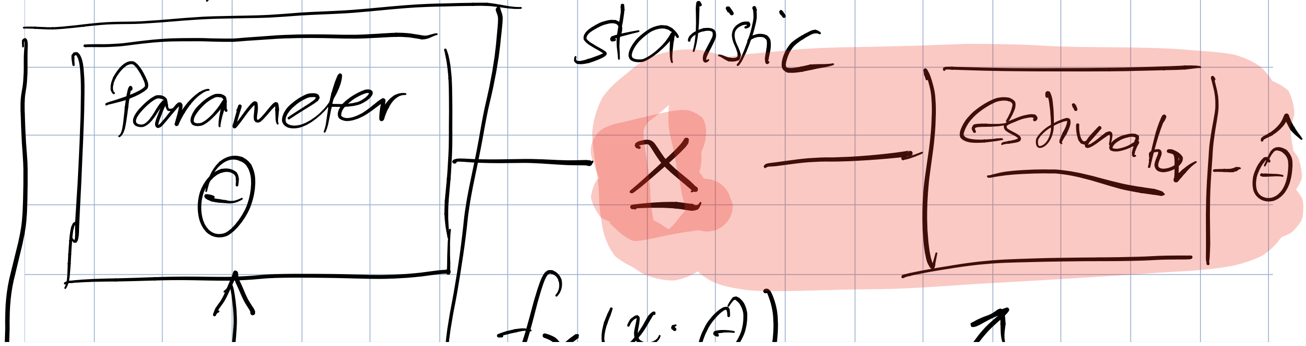
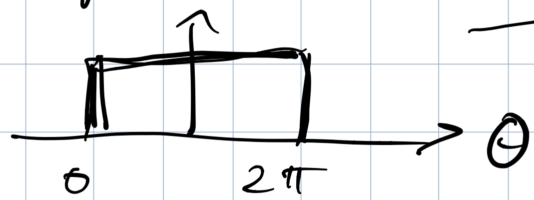
- CRLB theoretical tool
  bound — if close to bound ✓
- ideas estimator design
  identify optimal estimator ✓
- no design methodology ✗

Bayesian Estimation





prior information
a priori PDF



$f_{\underline{X}}(\underline{x};\theta) \longrightarrow$ parametarized PDF

$f_{\underline{X},\Theta}(\underline{x},\theta) \longrightarrow$ joint PDF

Bayes' Rule

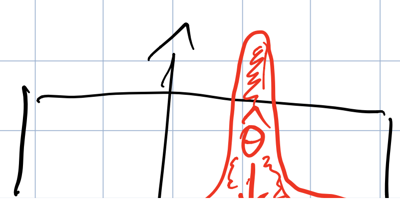
$$f_{\Theta|\underline{X}}(\theta|\underline{x}) = \frac{f_{\underline{X},\Theta}(\underline{x},\theta)}{f_{\underline{X}}(\underline{x})}$$

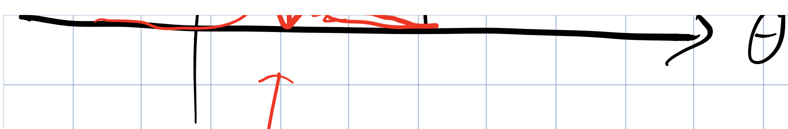
given observation

joint PDF

marginal PDF

$f_\Theta(\theta)$





$f_{\Theta|\underline{x}}(\theta|\underline{x})$

Likelihood fctn in θ

$$f_{\Theta|\underline{x}}(\theta|\underline{x}) = f_{\underline{x}|\Theta}(\underline{x}|\theta)\cdot f_{\Theta}(\theta)\,\frac{1}{f_{\underline{x}}(\underline{x})}$$

maximum likelihood (ML)

standard

- A priori PDF modeling

neglect $\ast\, f(\theta)$

$$\hat{\Theta} = \arg\left[\max_{\Theta} f_{\Theta|\underline{x}}(\Theta|\underline{x})\right]$$

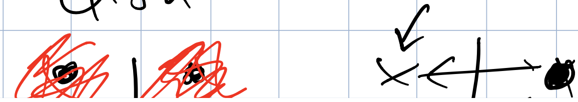
Minimum Mean-Square Error (MSSE)

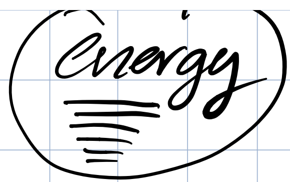
$$E\left((\hat{\Theta}-\Theta)^2\right)$$

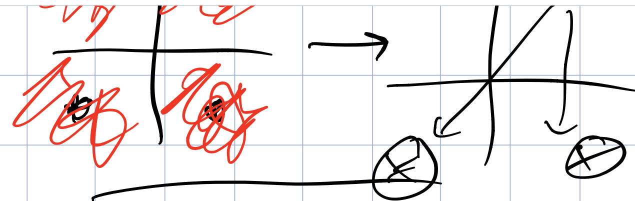
Detection:

QPSK

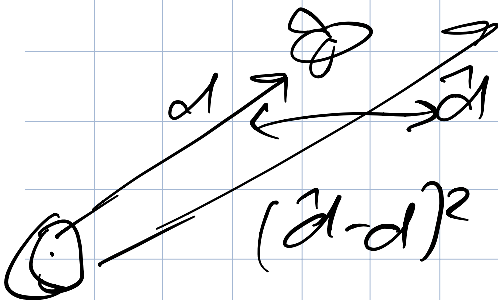
Error

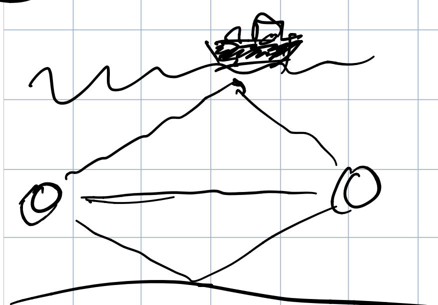


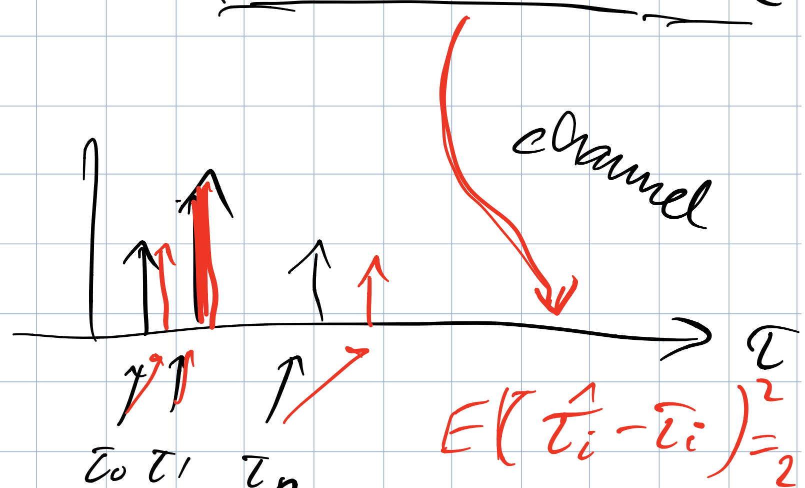




$$\Pr(\hat{\theta} \neq \theta) = \text{error rate}$$







$$E(\hat{\tau}_i - \tau_i)^2 = ?$$

MMSE:

minimize

$$E((\hat{\theta}-\theta)^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g(\underline{X}) - \theta)^2 f_{X|\Theta}(\underline{x}|\theta) \times f_\Theta(\theta)$$

$\hat{\theta}$

$$f_\Theta f_{X|\Theta} = f_{\Theta|X} \cdot f_X$$

marginal

$$= \int^{\infty} f_X(x) \int^{\infty} (g(X) - \theta)^2 f_{\Theta|X}(\theta|x)$$

$$\int_{-\infty} f_{\underline{x}} \dots \int_{-\infty} g^2 \dots f_{\theta|\underline{x}}(\theta|\underline{x})$$

need to minimize

$$g^2(\underline{x}) - 2\theta g(\underline{x}) + \theta^2$$

$$= \int_{-\infty}^{\infty} f_X(\underline{x})\, g^2(\underline{x}) \qquad \geq 0$$

$$-2\int_{-\infty}^{\infty} f_{\underline{x}}(\underline{x}) \int_{-\infty}^{\infty} \theta\, g(\underline{x})\, f_{\theta|\underline{x}}(\theta|\underline{x})$$

$\int \theta^n f(\theta) = E(\theta^n)$

$$+ \underbrace{\int_{-\infty}^{\infty} \theta^2 f_{\theta|\underline{x}}(\theta|\underline{x})}_{\cancel{Var(\theta|\underline{x})}} \qquad \geq 0$$

$$\int_{-\infty}^{\infty} f_X(x) \left(g^2(\underline{x}) - 2 \underbrace{\int_{-\infty}^{\infty} \theta\, g(x)\, f_{\theta|x}(\theta|\underline{x})} \right)$$

$$-2g(x)\underbrace{\int_{-\infty}^{\infty}\theta\, f_{\theta|\underline{x}}(\theta|\underline{x})}_{E(\theta|\underline{x})}$$

$$= \int_{-\infty}^{\infty} f_x(\underline{x})\left(g^2(\underline{x}) - 2g(\underline{x})E(\theta|\underline{x})\right.$$

$$\left. + E^2(\theta|\underline{x}) \underbrace{- E^2(\theta|\underline{x})}\right)$$

$$= \int_{-\infty}^{\infty} f_{\underline{x}}(\underline{x})\left(g(\underline{x}) - E(\theta|\underline{x})\right)^2 + \underbrace{E[\theta^2|\underline{x}] - E^2(\theta|\underline{x})}_{E(\theta^2) - E^2(\theta) = var(\theta)}$$

$$= \underbrace{\int_{-\infty}^{\infty} f_{\underline{x}}(\underline{x})\left(g(\underline{x}) - E(\theta|\underline{x})\right)^2}_{\text{minimize}} + \underbrace{var(\theta|\underline{x})}_{\text{cannot change}}$$

Estimator $\underline{X} \longrightarrow \boxed{g(\underline{X})} \longrightarrow \hat{\Theta}$

Choose MSSE estimator: $g(\underline{X}) = E(\Theta \mid \underline{X})$

conditional expectation

$$\min_{g} E\left[\left(g(\underline{X}) - \Theta\right)^2\right] = \mathrm{var}(\Theta \mid \underline{X})$$

best estimator to choose

best possible performance