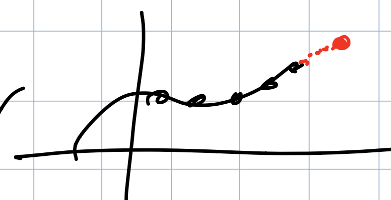
Linear Estimator

$$\hat{\Theta} = \sum_{i=1}^{n} a_i x_i + b$$

single

$\Rightarrow$ Linear Predictor 

vector parameter $\underline{\Theta} = (\Theta_1, \Theta_2 \cdots \Theta_p)$

$$p \left\{ \hat{\Theta}_j = \sum_{i=1}^{n} a_{ji} x_i + b_j \right. \qquad 1 \le j \le p$$

Collect p equations in

coefficients

n measurements

$$\begin{bmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \\ \vdots \\ \hat{\Theta}_p \end{bmatrix} = \hat{\underline{\Theta}} = A\underline{x} + \underline{b}$$

$$\underline{b} = E(\underline{\Theta}) - A^T \underline{\mu}_x$$

general linear predictor

A is $p \times n$:

Coefficients via partials

$$A = \begin{bmatrix} — \ \underline{a}_1^T \ — \\ — \ \underline{a}_2^T \ — \\ \vdots \\ — \ \underline{a}_p^T \ — \end{bmatrix}$$

$$\frac{d}{d\underline{a}_j} E\left[\left(\theta_j - \underline{a}_j^T \underline{x}\right)^2\right] \Rightarrow \underline{a}_j^T$$

$[a_{j1}, a_{j2} \ldots]$

$$\underline{a}_j^T = C_{\theta_j \underline{x}} \, C_{\underline{x}\underline{x}}^{-1}$$

$$E\left[\cancel{2}\left(\theta_j - \underline{a}_j^T \underline{x}\right) \cdot x_i\right] = 0 \quad ; \quad 1 \le i \le n$$

$$E[\theta_j x_i] - \underline{a}_j^T \cdot E[\underline{x} x_i] = 0$$ jth equ.

line up n equations $1 \le i \le n$

$\forall j$

$$E[\theta_1 x_1] = a_1^T \cdot E[\underline{x} \cdot x_1]$$

$$\vdots$$

$$E[\theta_1 x_n] = \underline{a}_1^T \, E[\underline{x} \cdot x_n]$$

$$\underbrace{\left[E[\theta_1 x_1], \cdots E[\theta_1 x_n] \right]}_{C_{\theta_1,\underline{x}}} = \underline{a}_1^T \cdot \underbrace{\begin{bmatrix} | & & | \\ E[\underline{x} x_1] & \cdots & E[\underline{x} x_n] \\ | & & | \end{bmatrix}}_{C_{xx}}$$

$$C_{\theta_1,\underline{x}} = \underline{a}_1^T \cdot C_{xx}$$

$$\boxed{\underline{a}_1^T = C_{\theta_1,\underline{x}} \cdot C_{xx}^{-1}}$$

$$\underline{a}_j^T = C_{\theta_j,\underline{x}} \cdot C_{\underline{x}\underline{x}}^{-1}$$

share $C_{\underline{x}\underline{x}}$

$C_{\theta_j,\underline{x}}$ for each j

independent linear

single-parameter estimator

$$\hat{\Theta}_j = \underline{a}_j^T \cdot \underline{x} + b_j$$

each parameter has its own estimator

$$\hat{\Theta} = \begin{bmatrix} \hat{\Theta}_1 \\ \vdots \\ \hat{\Theta}_p \end{bmatrix} = \begin{bmatrix} C_{\Theta_1 \underline{x}} \\ C_{\Theta_1 \underline{x}} \\ \vdots \\ C_{\Theta_p \underline{x}} \end{bmatrix} C_{xx}^{-1} \cdot \underline{x} + \begin{bmatrix} b_1 \\ \vdots \\ b_p \end{bmatrix}$$

$p \times n$

$$\hat{\underline{\Theta}} = C_{\underline{\Theta}, \underline{x}} \, C_{\underline{x}\underline{x}}^{-1} \cdot \underline{x} + \underline{b}$$

② linear Estimator + minimize $(\hat{\Theta}-\Theta)^2$

$\Rightarrow A : p \times n$ coefficient matrix

linear Minimum Mean-Square Error Estimator (LMMSE)

Optimal Bayesian +

flashback:

(1) Gaussian

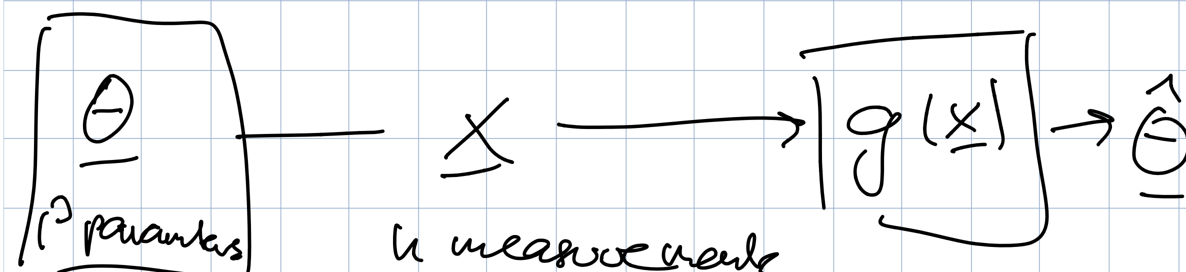
$$E[\underline{\theta}|\underline{x}] = C_{\underline{\theta},\underline{x}} \cdot C_{\underline{xx}}^{-1} \cdot \underline{x} \quad + \underline{b}$$

under

$[\underline{\theta}, \underline{x}]$ joint Gaussian $\Rightarrow$ error

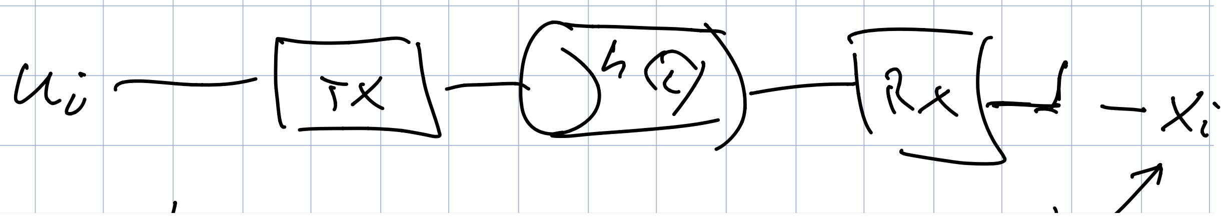
$$E[(\hat{\underline{\theta}}-\underline{\theta})(\hat{\underline{\theta}}-\underline{\theta})^T]$$

$$MSE_{\theta} = C_{\underline{\theta\theta}} - C_{\underline{\theta x}} C_{\underline{xx}}^{-1} C_{\underline{x\theta}}$$



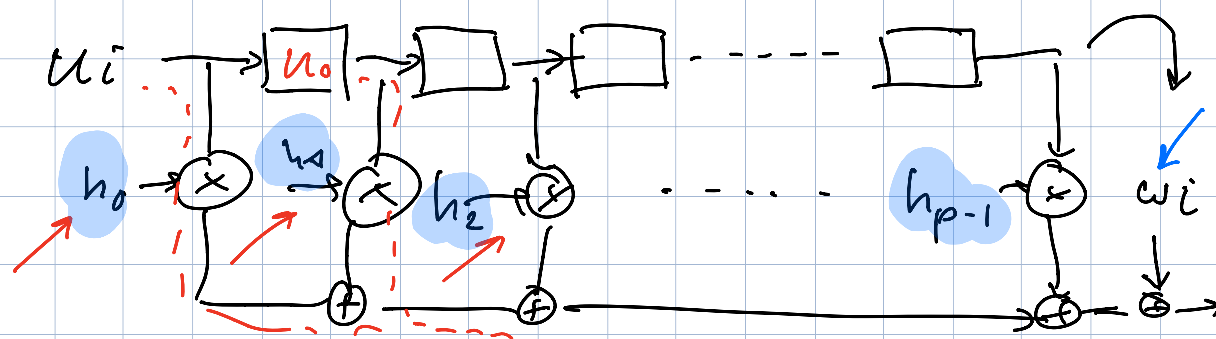
Block processing !

Channel Estimation



measurements

discrete FIR model



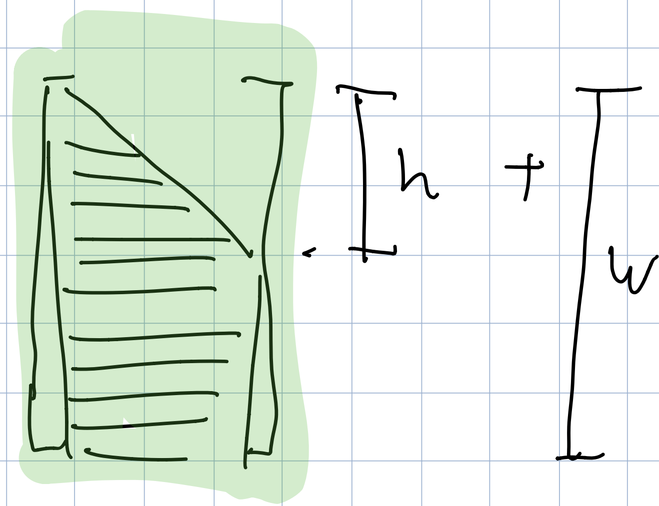
$$\underbrace{[h_0, h_1, \dots, h_{p-1}]}_{\text{unknown parameters}} \Longleftrightarrow h(\tau),\ H(f)$$

unknown parameters → fading Gaussian

Transmitting:

$$[u_0, u_1, u_2, \dots\dots]$$

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underline{X} = \begin{array}{c} \\ n \left\{ \right. \end{array} \overbrace{\begin{bmatrix} u_0 & \text{—} & & & \\ u_1 & u_0 & & & \\ u_2 & u_1 & u_0 & & \\ u_3 & u_2 & u_1 & u_0 & \\ \vdots & & & & \\ u_{n-1} & u_{n-2} & \cdots & \cdots & u_{n-p+1} \end{bmatrix}}^{p} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{p-1} \end{bmatrix}$$

$\underline{X} =$ 

parameters!

$$\underline{X} = \underset{n \times p}{U} \cdot \underline{h} + \underline{w}$$

$$\hat{\underline{h}} = C_{\underline{h}\cdot\underline{X}} \cdot C_{\underline{X}\underline{X}}^{-1} \cdot \underline{X}$$

$$C_{XX} = E[\underline{X} \cdot \underline{X}^T] \quad E\begin{bmatrix} x_1x_1 & x_{12} & \cdots \\ & x_2x_2 & \\ \cdots & & \end{bmatrix}$$

$$= E[(U\underline{h}+\underline{w})(U\underline{h}+\underline{w})^T]$$

$$= E[(U\underline{h}+\underline{w})(\underline{h}^T U^T + \underline{w}^T)]$$

$$= E[U \underbrace{\underline{h}}\, \underline{h}^T \cdot U^T] + E[\underline{w}\,\underline{w}^T]$$

$$= U\, E[\underline{h}\,\underline{h}^T]\, U^T + \sigma^2 I$$

$$C_{\underline{h}\underline{x}} = E[\underline{h}\underbrace{(U\underline{h} + \underline{w})^T}_{\underline{x}^T}] \qquad I_n$$

$$C_{\underline{h}\underline{x}} = E[\underline{h}\,\underline{h}^T \cdot U^T] = \underbrace{E[\underline{h}\,\underline{h}^T]}_{C_{hh}}\, U^T$$

$$\boxed{\hat{h} = C_{hh} \cdot U^T \left(U\, C_{hh}\, U^T + \sigma^2 I\right)^{-1} \cdot \underline{x}}$$

$$C_{hh} = \{E[h_i h_j]\}$$

$E[h_i h_j] = 0 \quad i \neq j$

usually not true

very special case !!!

$$\Rightarrow C_{nn} \Rightarrow \sigma_n^2 \cdot I$$

$$\hat{h} = \sigma_n^2 \cdot U^T \left(U U^T \cdot \sigma_n^2 + \sigma^2 I \right)^{-1} \underline{x}$$

$$= \cancel{\sigma_n^2}\, U^T \left(\cancel{\sigma_n^2} \left(U U^T + \frac{\sigma^2}{\sigma_n^2} I \right) \right)^{-1} \cdot x$$

$$\hat{h} = U^T \left(U U^T + \frac{\sigma^2}{\sigma_n^2} I \right)^{-1} \cdot \underline{x}$$

$$\begin{array}{cc} & u_j \\ & \begin{bmatrix} \mid & U^T \end{bmatrix} \\ u_j^T & \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \end{array} \approx I$$

$$\underline{u}_j^T \cdot \underline{u}_j = 1 \Rightarrow \text{unit energy}$$

$$\underline{u}_j^T \cdot \underline{u}_{j\pm\delta} = 0$$

$$\underline{u} = [\, u_1\, u_2 \text{ ------- } u_n \,]$$

$$\underbrace{\qquad\qquad\qquad}_{u_j\, u_{j+1} \,\cdots\, u_{j+p-1}}$$

$$\left| u_{j-1}\, u_j\, u_{j+1} \text{ -----} \right| \approx 0$$

if u are i.i.d. and $P \to \infty$

$$u_i^T u_{j+\delta} \longrightarrow \frac{c}{\sqrt{P}}$$

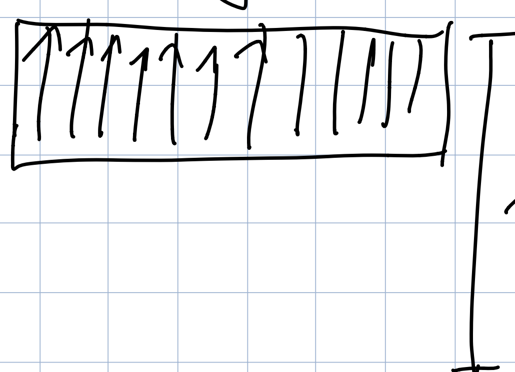
$$\hat{h} = U^T \left(I + \frac{\sigma^2}{\sigma_h^2}\right)^{-1} \cdot \underline{x}$$

correlation

$$\boxed{\hat{\underline{h}} = \frac{\sigma_h^2}{\sigma^2 + \sigma_h^2} \cdot U^T \cdot \underline{x}}$$

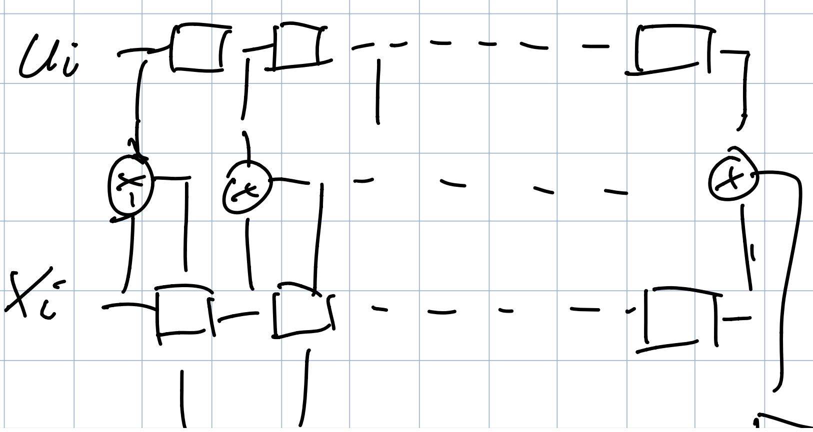
≈ optimal

matlab

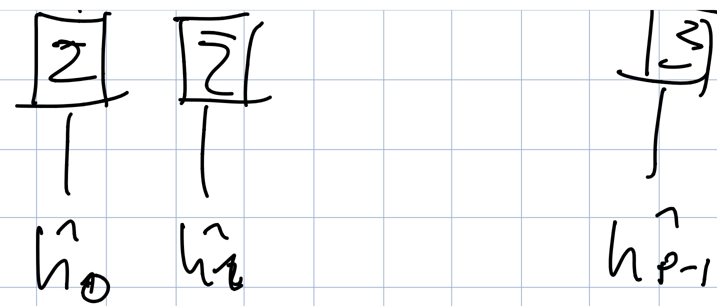


$\underline{x} \to \hat{\underline{h}}$

Receiver:



simulink



- exploit channel cross correlations
- $h_p \longrightarrow h_p(t)$

  Kalman

- batch processing

$\Downarrow$

refinements