Kalman Filter (1960)

$$\underline{X} = U \underline{\theta} + \underline{w}$$

Canonical estimation problem

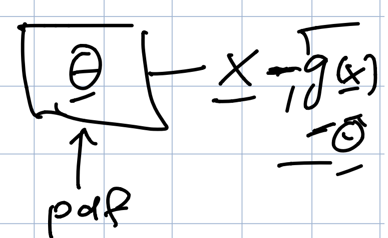
$$X[n] = \underline{U}^T[n] \cdot \underline{\theta} + w[n]$$

$$= U^T[n] \cdot \underline{s}[n] + w[n]$$

- p parameters
- now: they change with n

How do approach the dynamics of the parameter space.

1) Fixed parameter



2) linear predictor

$$\Theta \longrightarrow x[n] \Rightarrow \hat{x}[n] = f(\underline{x})$$

causal requirement

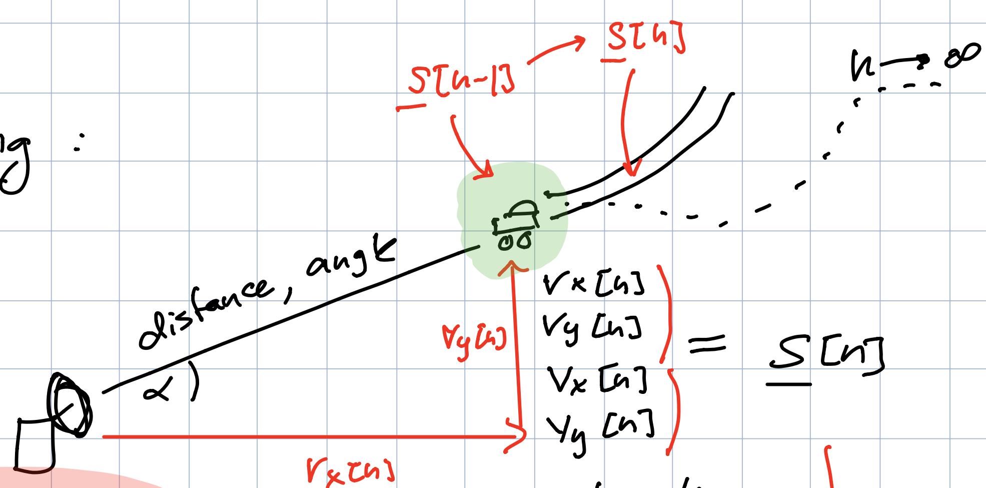
dependence

$$p(\underline{x}, x[n]) \neq p(\underline{x})\, p(x[n])$$

3) $\underline{\Theta} \longrightarrow \underline{s}[n]$

describes an internal state

Tracking :



$$\begin{pmatrix} v_x[n] \\ v_y[n] \\ r_x[n] \\ r_y[n] \end{pmatrix} = \underline{s}[n]$$

non-stationary

- acceleration
- road
- ⋮

- State-space vector

General Model :

$$\underline{s}[n] = A\, \underline{s}[n-1] + B\, \underline{u}[n] + K \cdot \underline{v}[n]$$

i.i.d. $\sim \mathcal{N}(0, \sigma^2)$

$$\underline{s}[n] = f(\underline{s}[n-1])$$

- uncertainty

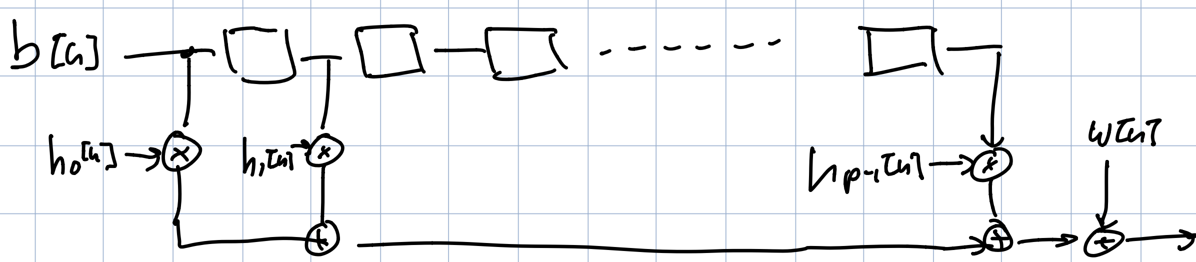
Unscented Kalman

- external input

Vehicle tracking:

$$\underline{s}[n] = \begin{bmatrix} r_x[n] \\ r_y[n] \\ v_x[n] \\ v_y[n] \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} r_x[n-1] \\ r_y[n-1] \\ v_x[n-1] \\ v_y[n-1] \end{bmatrix}}_{\underline{s}[n-1]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ u_x[n] \\ u_y[n] \end{bmatrix}}_{\underline{u}[n]}$$

Example: Unknown Channel



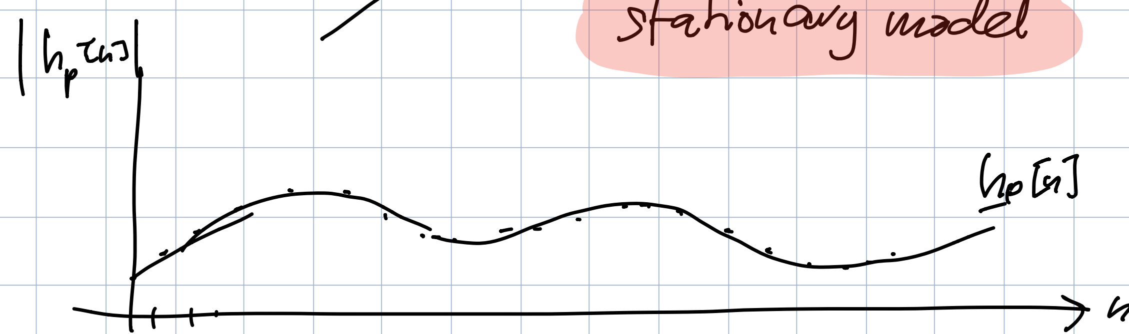
State: $\underline{h}[n] = \begin{bmatrix} h_0[n] \\ h_1[n] \\ \vdots \\ h_{p-1}[n] \end{bmatrix}$

how do the h's evolve?

complex numbers

$$\underline{h}[n] = A\,\underline{h}[n-1] + B\,\underline{u}[n]$$

stationary model



$$A = \begin{bmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_{p-1} \end{bmatrix} + \begin{bmatrix} \sqrt{1-\alpha_0} & & & \\ & \sqrt{1-\alpha_1} & & \\ & & \ddots & \\ & & & \sqrt{1-\alpha_{p-1}} \end{bmatrix} \underline{u}[n]$$

α_p : forgetting factors < 1

$$\underline{h}[n] \sim N(0,1)$$

Kalman Measurement model:

$$x[n] = \underline{u}^T[n]\,\underline{s}[n-1] + w[n]$$

multiple measurements

Measurement Model

P parameters

$$\underline{x}[n] = U[n]\cdot\underline{s}[n] + Q\,\underline{w}[n]$$

M measurements

M×P matrix

State-space model

$$\underline{s}[n] = A[n] \cdot \underline{s}[n-1] + B \cdot u[n]$$

general formula

1) step $\quad \hat{\underline{x}}[n] = C_{\underline{x}[n] \cdot \underline{x}} \; C^{-1}_{\underline{x}\underline{x}} \cdot \underline{x}$

M measurement

mean values neglected!

$$\underline{x}[n] = U \cdot \underline{s}[n] + Q\,\underline{w}[n]$$

$$= \quad \underline{x}[n] = U \cdot A \cdot s[n-1] + \cancel{Q\,\underline{w}[n]} + U\,\cancel{B} \cdot u[n]$$

$$\hat{\underline{x}}[n] = \underbrace{U \cdot A \cdot C_{\underline{s}[n-1] \cdot \underline{x}} \; C^{-1}_{\underline{x}\underline{x}} \; \underline{x}}$$

all previous measurements

cross-covariance parameter – data

data autocovariance

$$\hat{\underline{x}}[n] = U \cdot A \cdot \underbrace{\hat{\underline{s}}[n-1]}$$

not only along $\underline{x}$

only $\hat{s}[n-1]$, $\hat{x}[n-1]$

$$\hat{\underline{x}}[n] = U \cdot \hat{\underline{s}}[n|n-1]$$

predicted state @ n

given the MMSE estimate

given the MMSE estimate $\hat{s}[n-1]$

$$\hat{s}[n] = \underbrace{\hat{s}[n|n-1]}_{A\cdot\hat{s}[n-1]} + \underline{k}[n]\underbrace{(\underline{x}[n]-\hat{\underline{x}}[n])}_{\text{prediction error}}$$

$$\hat{s}[n] = \hat{s}[n|n-1] + \underline{k}[n]\left(\hat{\underline{X}}[n] - U\cdot\hat{s}[n|n-1]\right)$$

Kalman Equations

additional w. dynamic parameters

State Prediction: $\hat{s}[n|n-1] = A\hat{s}[n-1]$

MSE Prediction: $M[n|n-1] = A M[n-1] A^T + BB^T$

estimate @ time n

error covariance of state @ n-1

Gain Update: $$\underline{k}[n] = \frac{M[n|n-1]\,U[n]}{U^T[n]\,M[n|n-1]\,U[n] + \sigma^2}$$

Estimator Update:

$$\hat{\underline{s}}[n] = \hat{\underline{s}}[n|n-1] + \underline{K}[n]\left(\underline{x}[n] - U\underline{s}[n|n-1]\right)$$

Error Variance:

$$M[n] = \left(\underline{I} - \underline{K}[n]\,U[n]\right)M[n|n-1]$$

Vehicle Tracking:

$B = I$

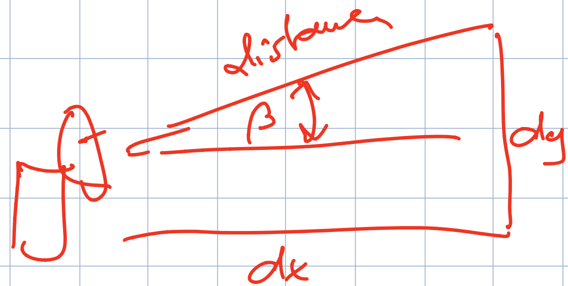
$$\underline{s}[n] = \underbrace{\begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} r_x[n-1] \\ v_y[n-1] \\ v_x[n-1] \\ v_y[n-1] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ u_x[n] \\ u_y[n] \end{bmatrix}}_{\text{innovation}}$$

Linear Model:

$Q = I$

$$\underbrace{\begin{bmatrix} d_x[n] \\ d_y[n] \end{bmatrix}}_{\underline{x}[n]} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{U} \underline{s}[n] + \begin{bmatrix} w_x[n] \\ w_y[n] \end{bmatrix}$$

⇒ often non-linear observations:



$$\underline{x}[n] = \begin{bmatrix} \beta[n] \\ R[n] \end{bmatrix} = \underline{h}(\underline{s}[n]) + \underline{w}[n]$$

$$h[\underline{s}] = \begin{bmatrix} \tan^{-1}\left(v_y[n]/v_x[n]\right) \\ \sqrt{v_x^2[n]+v_y^2[n]} \end{bmatrix} \neq U\underline{s}+\underline{w}$$

linearize .....

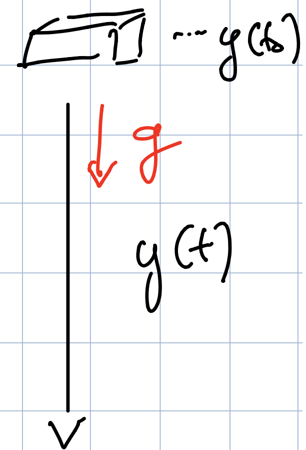
Extended Kalman Filtering

- tracking ⟶ toy example
- channel tap (gains) ⟶ R&D
- Physical dynamics

doppler estimation
for sat comm

Falling Body:

$$\ddot{y}(t) = -g \qquad \text{acceleration}$$



$$\dot{y}(t) = \dot{y}(t_0) - g\,(t-t_0)$$

starting

$\Rightarrow$ Integrating velocity over time

$$y(t) = y(t_0) + \dot{y}(t_0)\,(t-t_0) - \frac{g}{2}(t-t_0)^2$$

$\Rightarrow$ discretization: time in multiples of ΔT

$$y[n] = y[n-1] + \dot{y}[n-1]\,\Delta T - \frac{g}{2}\Delta T^2$$

$y(n\Delta T)$

$\left[\, y[n] \,\right]$

State:

$$\underline{s}[n] = \begin{bmatrix} y[n] \\ \dot{y}[n] \end{bmatrix}$$

"Physics"

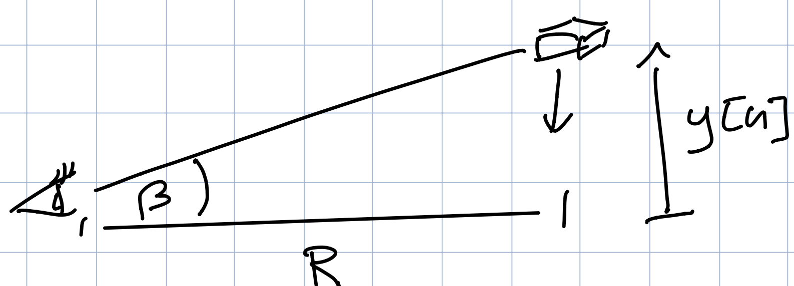
$$\underline{s}[n] = \begin{bmatrix} 1 & \Delta T \\ 0 & 1 \end{bmatrix} \underline{s}[n-1] - g \underbrace{\begin{bmatrix} \frac{\Delta T^2}{2} \\ \Delta T \end{bmatrix}}_{\text{control}}$$

$$\dot{y}[n] = \dot{y}[n-1] - g\,\Delta T$$

$$y[n] = y[n-1] + \Delta T\,\dot{y}[n-1] - \frac{g}{2}\Delta T^2$$

Measurement Model:

"Engineering"



$$\beta[n] = \tan^{-1}\left(\frac{y[n]}{R}\right) = h(\underline{s}[n]) + w[n]$$

$$R \longrightarrow \infty$$

$$\approx y[n]/R + w[n]$$

linear

$$\beta[n] = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \underline{s}[n] + w[n]$$