Terza Prova Intermedia di ANALISI MATEMATICA, 12 gennaio 2005

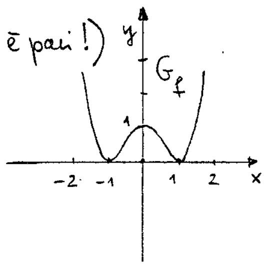
FILA A 1) $\displaystyle\int_0^1 \frac{x}{x^2+3}\,dx = \left[\frac{1}{2}\log(x^2+3)\right]_0^1 = \frac{1}{2}\left[\log 4 - \log 3\right] = \frac{1}{2}\log\frac{4}{3} = \underline{\underline{\log\sqrt{\frac{4}{3}}}}.$

$$\int_3^4 \frac{x^2+2x-8}{x-2}\,dx = \int_3^4 \frac{\cancel{(x-2)}(x+4)}{\cancel{x-2}}\,dx = \int_3^4 (x+4)\,dx = \left[\frac{x^2}{2}+4x\right]_3^4 = (8+16)-\left(\frac{9}{2}+12\right) = \underline{\underline{\frac{15}{2}}}.$$

$$\int_{-1}^1 \left(|\sqrt[3]{x}|+\sqrt[3]{x}\right)dx = \int_{-1}^{1}\left(-\sqrt[3]{x}+\sqrt[3]{x}\right)dx + \int_0^1\left(\sqrt[3]{x}+\sqrt[3]{x}\right)dx = 2\int_0^1\sqrt[3]{x}\,dx = 2\left[\frac{x^{\frac{4}{3}}}{4/3}\right]_0^1 = \underline{\underline{\frac{3}{2}}}.$$ $\square$

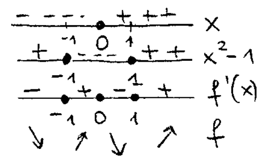
2) i) $f(x)=(x^2-1)^2$; $\operatorname{dom} f = \mathbb{R}$; $f(x)\geq 0 \ \forall x\in\mathbb{R}$ e $=0 \iff x=\pm 1$;

(f è pari!)



$\displaystyle\lim_{x\to-\infty} f(x) = +\infty \quad \lim_{x\to+\infty} f(x) = +\infty.$

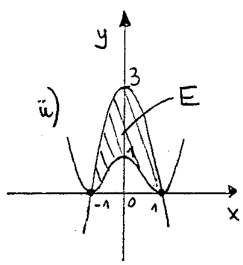
$\operatorname{dom} f' = \mathbb{R} \qquad f'(x) = 2(x^2-1)2x$



$x=-1$, $x=1$ sono pt. di min loc.
$f(-1)=f(1)=0$
(sono pt. di minimo globali)

$x=0$ pt. di massimo loc., $f(0)=1$.

ii)



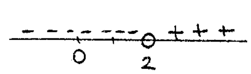
ii) area $E = \displaystyle\int_{-1}^1 (g(x)-f(x))\,dx = 2\int_0^1 (g(x)-f(x))\,dx =$

$= 2\displaystyle\int_0^1\left(-3x^2+3-x^4+2x^2-1\right)dx =$

$= 2\left[-\cancel{3}\dfrac{x^3}{\cancel{3}}+3x-\dfrac{x^5}{5}+2\dfrac{x^3}{3}-x\right]_0^1 = 2\left[-1+3-\dfrac{1}{5}+\dfrac{2}{3}-1\right]$

$= \underline{\underline{\dfrac{44}{15}}}.$ $\square$

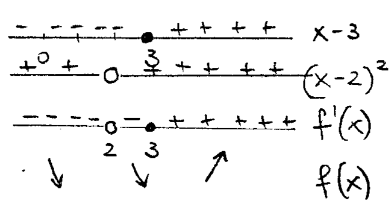
3) $f(x)=\dfrac{e^x}{x-2}$: i) $\operatorname{dom} f = \mathbb{R}\setminus\{2\}$



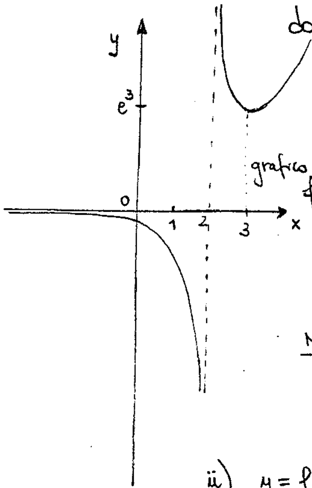
$\displaystyle\lim_{x\to-\infty} f(x) = 0 \quad \lim_{x\to 2^-} f(x) = -\infty \quad \lim_{x\to 2^+} f(x) = +\infty \quad \lim_{x\to+\infty} f(x) = +\infty.$

$\operatorname{dom} f' = \mathbb{R}\setminus\{2\} \qquad f'(x) = \dfrac{e^x(x-2)-e^x}{(x-2)^2} = \dfrac{e^x(x-3)}{(x-2)^2}$

$f'(x)=0 \iff x=3.$



$x=3$ pt. di min. loc. $f(3)=e^3$



$\mathrm{dom} f'' = \mathbb{R}\setminus\{2\}$ $f''(x) = \dfrac{\left(e^x(x-3)+e^x\right)(x-2)^{\cancel{2}} - e^x(x-3)\,2\,\cancel{(x-2)}}{(x-2)^{\cancel{4}\,3}}$

$$= \frac{e^x\left((x-3)(x-2) + x-2 - 2(x-3)\right)}{(x-2)^3}$$

$$= \frac{e^x\left(x^2-5x+6+x-2-2x+6\right)}{(x-2)^3} = \frac{e^x\left(x^2-6x+10\right)}{(x-2)^3}$$

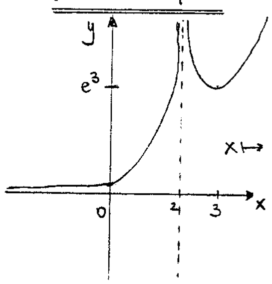
Nota: $e^x(x^2-6x+10)>0$ $\forall x\in\mathbb{R}$

$- - - - - - -$	$\underset{2}{\circ}$	$+ + + +$	$f''(x)$
$\cap$		$\cup$	$f(x)$

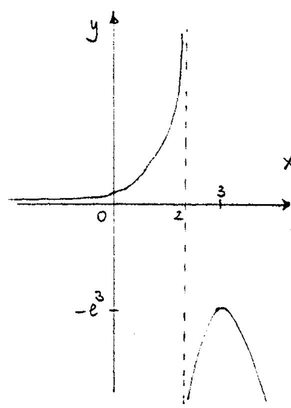
ii) $y = f(0) + f'(0)(x-0)$ Nota: $f'(0) = \dfrac{-3}{4}$

$$\underline{\underline{y = -\tfrac{1}{2} - \tfrac{3}{4}x}}$$

iii)



$x \mapsto |f(x)|$



$x \longmapsto -f(x)$.

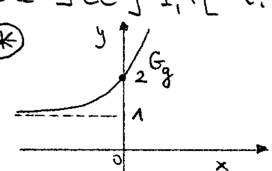
Grafici approssimativi!

iv) $\displaystyle\int_0^1 e^{-x} f(x)\,dx = \int_0^1 \frac{1}{x-2}\,dx =$

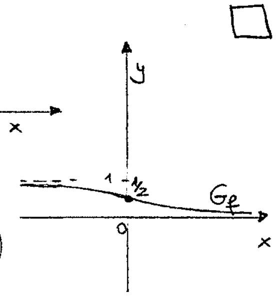
$$= \Big[\log|x-2|\Big]_0^1 = \log 1 - \log 2 = \underline{\underline{\log\tfrac{1}{2}}}.$$ $\square$

4) Poniamo $f(x) = e^x + \sqrt[3]{x} - \dfrac{1}{x^2+1}$; f è una funzione continua su $[-1,1]$ (anzi, su tutto $\mathbb{R}$).
Inoltre $f(-1) = e^{-1} - 1 - \frac{1}{2} = \frac{1}{e} - \frac{3}{2} < 0$, $f(1) = e+1-\frac{1}{2} = e+\frac{1}{2} > 0$.
Per il teorema di esistenza degli zeri segue che $\exists\, c\in\,]-1,1[$ t.c. $f(c)=0$, ossia esiste almeno una soluzione dell'eq. data. ⊛ $\square$

5) Sia $g(x) = e^x+1$. Il suo grafico è



da cui si ha rapidamente il grafico di f



(si ottiene anche studiando la funzione $\dfrac{1}{e^x+1}$ direttamente)

i) (F) f ha massimo in $[0,+\infty[$, ma non ha minimo.

ii) (V) e^x+1 è crescente; $\dfrac{1}{e^x+1}$ è decrescente.

⊛ In questo caso si vede anche immediatamente che $x=0$ è soluzione dell'eq. !!!

iii) $\int_{-1}^{3} f(x)\,dx \geq 0$ (V) essendo $f(x) \geq 0$;

$\int_{0}^{1} f(x)\,dx \leq 1$ (V) poiché $f(x) \leq 1$ su $[0,1]$ per cui

$$\int_0^1 f(x)\,dx \leq \int_0^1 1\,dx = 1 .$$

iv) (V) poiché $f(x) = \frac{1}{e^x+1} \leq \frac{1}{e^x} = e^{-x}$ $\forall x \in \mathbb{R}$.

$$\sum_{k=0}^{4} f(k) = \frac{1}{e^0+1} + \frac{1}{e^1+1} + \frac{1}{e^2+1} + \frac{1}{e^3+1} + \frac{1}{e^4+1} = \frac{1}{2} + \frac{1}{e+1} + \frac{1}{e^2+1} + \frac{1}{e^3+1} + \frac{1}{e^4+1} .$$ □

6) $P_8^{6,2} = \frac{8!}{6!\,2!} = \frac{\cancel{8}^{\,4} \cdot 7 \cdot \cancel{6!}}{\cancel{6!}\,\cancel{2}} = \underline{\underline{28}}$. ■

FILA (B) 1) $\int_{-1}^{1} \frac{x^2-x-12}{x+3}\,dx = \int_{-1}^{1} \frac{\cancel{(x+3)}(x-4)}{\cancel{x+3}}\,dx = \int_{-1}^{1} (x-4)\,dx = \left[\frac{x^2}{2} - 4x\right]_{-1}^{1} = \left(\frac{1}{2}-4\right) - \left(\frac{1}{2}+4\right) = \underline{\underline{-8}}$.

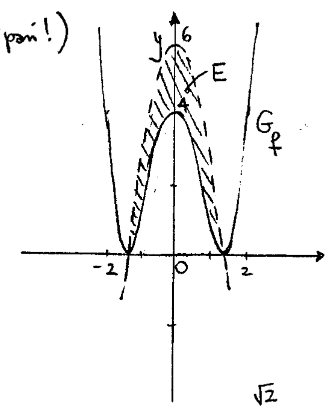
$\int_0^1 x e^{x^2+1}\,dx = \frac{1}{2}\left[e^{x^2+1}\right]_0^1 = \underline{\underline{\frac{1}{2}\left(e^2 - e\right)}}$.

$\int_{-1}^{1} \left(|\sqrt[5]{x}| - \sqrt[5]{x}\right) dx = \int_{-1}^{0}\left(-\sqrt[5]{x} - \sqrt[5]{x}\right) dx + \int_{0}^{1}\left(\sqrt[5]{x} - \sqrt[5]{x}\right) dx = -2\int_{-1}^{0} \sqrt[5]{x}\,dx = -2\left[\frac{x^{\frac{6}{5}}}{\frac{6}{5}}\right]_{-1}^{0} = \underline{\underline{\frac{5}{3}}}$

□

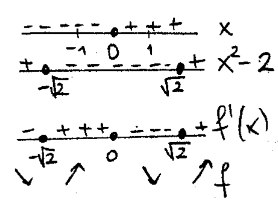
2) i) $f(x) = (x^2-2)^2$; dom $f = \mathbb{R}$, $f(x) \geq 0$ $\forall x \in \mathbb{R}$ e $=0 \iff x = \pm\sqrt{2}$;

(f è pari!)



$\lim_{x \to -\infty} f(x) = +\infty$ $\lim_{x \to +\infty} f(x) = +\infty$.

dom $f' = \mathbb{R}$ $f'(x) = 2(x^2-2)\,2x$



$x = -\sqrt{2}$, $x = \sqrt{2}$ pt. di min loc.
$f(-\sqrt{2}) = f(\sqrt{2}) = 0$
(sono pt. di min. globali)

$x = 0$ pt. di max loc. $f(0) = 4$

ii) area $E = \int_{-\sqrt{2}}^{\sqrt{2}} (g(x) - f(x))\,dx = 2\int_0^{\sqrt{2}} (g(x) - f(x))\,dx = 2\int_0^{\sqrt{2}} \left(-3x^2+6 - (x^2-2)^2\right) dx =$

$= 2\int_0^{\sqrt{2}} \left(-3x^2+6 - x^4 + 4x^2 - 4\right) dx = 2\int_0^{\sqrt{2}} \left(-x^4 + x^2 + 2\right) dx =$

$= 2\left[-\frac{x^5}{5} + \frac{x^3}{3} + 2x\right]_0^{\sqrt{2}} = 2\left(-\frac{4\sqrt{2}}{5} + \frac{2\sqrt{2}}{3} + 2\sqrt{2}\right) = \underline{\underline{\frac{56\sqrt{2}}{15}}}$.

3) $f(x)=\dfrac{e^x}{1-x}$: i) $\mathrm{dom}\, f = \mathbb{R}\setminus\{1\}$

$+ + + + \circ - - - - - -$ segno di f (0, 1)

$\lim_{x\to-\infty} f(x)=0 \qquad \lim_{x\to 1^-} f(x)=+\infty \qquad \lim_{x\to 1^+} f(x)=-\infty \qquad \lim_{x\to+\infty} f(x)=-\infty.$

$\mathrm{dom}\, f' = \mathbb{R}\setminus\{1\} \qquad f'(x)=\dfrac{e^x(1-x)+e^x}{(1-x)^2}=\dfrac{e^x(2-x)}{(1-x)^2}$

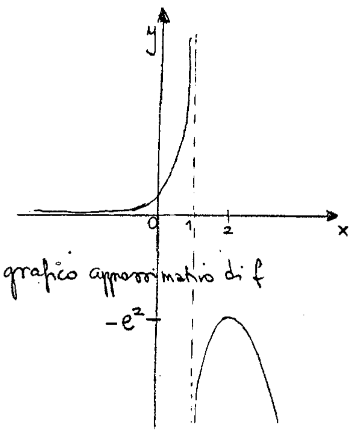


grafico approssimativo di f

$f'(x)=0 \iff x=2$

$+ + + + \bullet - - - - -$ $2-x$ (2)
$+ + \circ + + + +$ $(1-x)^2$ (1)
$+ + \circ + \bullet - - - - -$ $f'(x)$ (1, 2)
$\nearrow \quad \nearrow \quad \searrow$ $f(x)$

$x=2$ pt. di max loc. $f(2)=-e^2$.

$\mathrm{dom}\, f'' = \mathbb{R}\setminus\{1\} \qquad f''(x)=\dfrac{\left(e^x(2-x)-e^x\right)(1-x)^{\cancel{2}}+e^x(2-x)\,2\,\cancel{(1-x)}}{(1-x)^{\cancel{4}3}}$

$=\dfrac{e^x\left((2-x)(1-x)-1+x+4-2x\right)}{(1-x)^3}$

$=\dfrac{e^x\left(x^2-3x+2+3-x\right)}{(1-x)^3}=\dfrac{e^x\left(x^2-4x+5\right)}{(1-x)^3}$

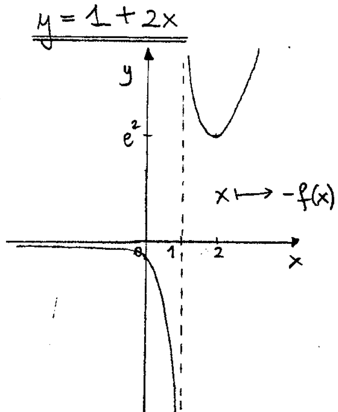
Nota: $e^x(x^2-4x+5)>0$ $\forall x\in\mathbb{R}$

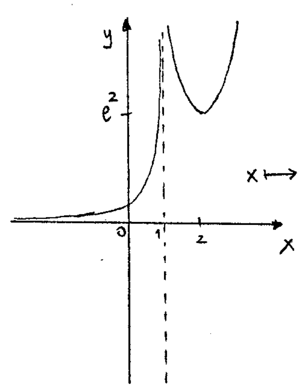
$+ + + + \circ - - - - -$ $f''(x)$ (0, 1)
$\cup \qquad \cap$ $f(x)$

ii) $y=f(0)+f'(0)(x-0) \qquad f'(0)=2$

$\underline{\underline{y=1+2x}}$

iii)





iv) $\displaystyle\int_2^3 \frac{f(x)}{e^x}\,dx=\int_2^3\frac{1}{1-x}\,dx=-\int_2^3\frac{1}{x-1}\,dx=-\Big[\log|x-1|\Big]_2^3=-(\log 2-\log 1)=\underline{\underline{\log\tfrac{1}{2}}}.$

$\square$

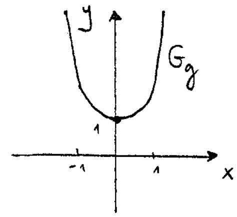
4) Poniamo $f(x)=\log x-|x-e|-\frac{1}{x}$; f è continua su $[1,e]$. Inoltre

$f(1)=\log 1-|1-e|-1=-(e-1)-1=-e<0$

$f(e)=\log e-\frac{1}{e}=1-\frac{1}{e}>0$. Per il teorema di esistenza degli zeri segue che

$\exists c\in\,]1,e[$ t.c. $f(c)=0$, ossia esiste almeno una soluzione dell'eq. data. $\square$

5) Sia $g(x)= x^4+1$. Il suo grafico è



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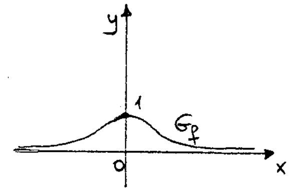

(si ottiene anche studiando la funzione $\frac{1}{x^4+1}$ direttamente)

i) (F) f ha massimo in $[0,3[$, ma non minimo.

ii) (F) f è crescente su $]-\infty,0]$ e decrescente su $[0,+\infty[$.

iii) $\int_{-1}^{3} f(x)\,dx \geq 0$ (V) essendo $f(x) \geq 0$,

$\int_{0}^{1} f(x)\,dx \leq 1$ (V) essendo $f(x) \leq 1$ su $[0,1]$ per cui $\int_{0}^{1} f(x)\,dx \leq \int_{0}^{1} 1\,dx = 1$.

iv) (F) : $\frac{1}{x^4}$ non è definita in $x=0$!

$$\sum_{k=0}^{3} f(k) = \frac{1}{1} + \frac{1}{1+1} + \frac{1}{2^4+1} + \frac{1}{3^4+1} = 1 + \frac{1}{2} + \frac{1}{17} + \frac{1}{82}.$$

□

6) agli estremi ci sono palline blu ; allora $P_{10}^{5,5} = \frac{10!}{5!\,5!}$

agli estremi ci sono palline rosse : allora $P_{10}^{7,3} = \frac{10!}{7!\,3!}$.

Abbiamo quindi $P_{10}^{5,5} + P_{10}^{7,3}$.