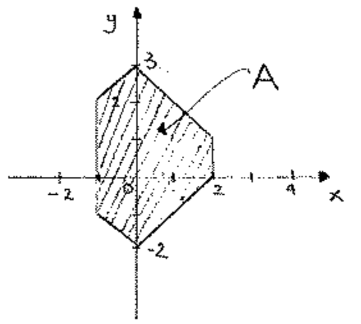
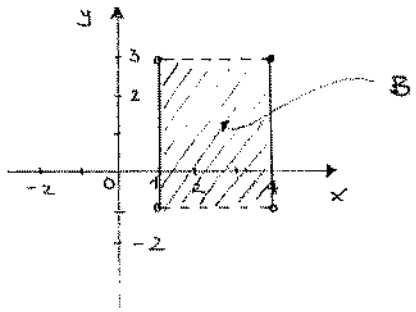
Esame scritto di ANALISI MATEMATICA — Rovereto, 7 settembre 2005

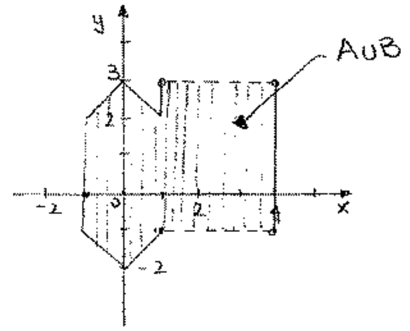
1) i) $A = \{(x,y)\in\mathbb{R}^2 : \; -1 \le x \le 2 \;,\; -2+|x| \le y \le -|x|+3\}$

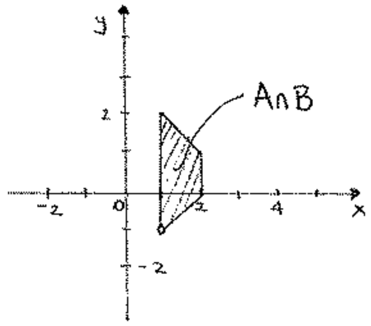
$B = [1,4] \times]-1,3[$.

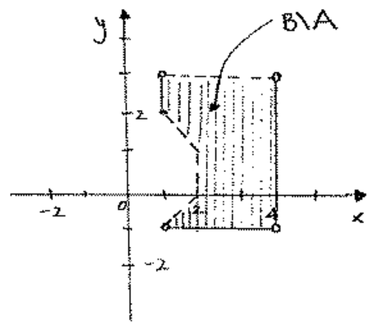




ii)





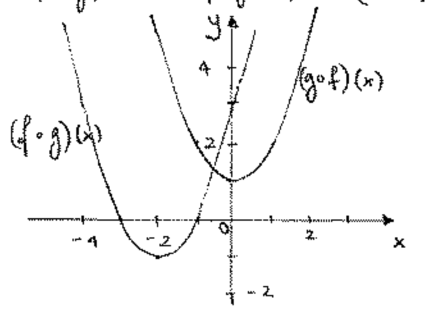


iii) area $(A\cup B) = 4\cdot 2 + 3\cdot 4 = \underline{20}$ area $(A\cap B) = \underline{2}$ area $(B\setminus A) = 4\cdot 3 - 2 = \underline{10}$.

2) i) $f(x) = x^2-1 \qquad g(x) = x+2$ in $\mathbb{R}$

$$(g\circ f)(x) = g(f(x)) = (x^2-1)+2 = x^2+1 \quad \forall x\in\mathbb{R}$$

$$(f\circ g)(x) = f(g(x)) = (x+2)^2 - 1 = x^2+4x+3 \quad \forall x\in\mathbb{R}.$$



ii)

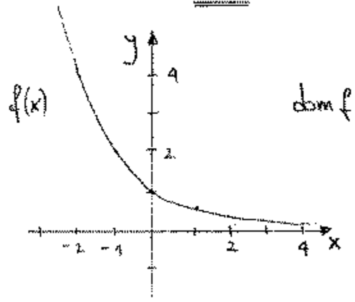
È falso che $(g\circ f)(x) = (f\circ g)(x) \;\; \forall x\in\mathbb{R}$: infatti, per $x=0$, $(g\circ f)(0) = 1$
$(f\circ g)(0) = 3$.

$$(g\circ f)(x) = (f\circ g)(x) \iff x^2+1 = x^2+4x+3 \iff 4x = -2 \Rightarrow x = -\frac{1}{2}.$$

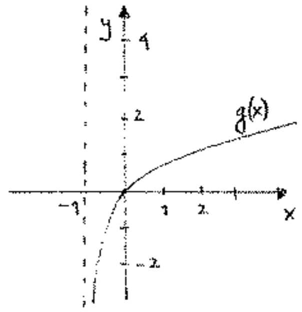
iii) $$\int_{-1}^{2} f(x)g(x)\,dx = \int_{-1}^{2}(x^2-1)(x+2)\,dx = \int_{-1}^{2}(x^3+2x^2-x-2)\,dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{2} =$$

$$\int_{-1}^{2}(f\circ g)(x)\,dx = \int_{-1}^{2}\left(x^2+4x+3\right)dx = \left[\frac{x^3}{3}+\cancel{4}\frac{x^2}{\cancel{2}}+3x\right]_{-1}^{2} = \left(\frac{8}{3}+8+6\right)-\left(-\frac{1}{3}+2-3\right)$$
$$= \underline{18}\,.$$

3) i)

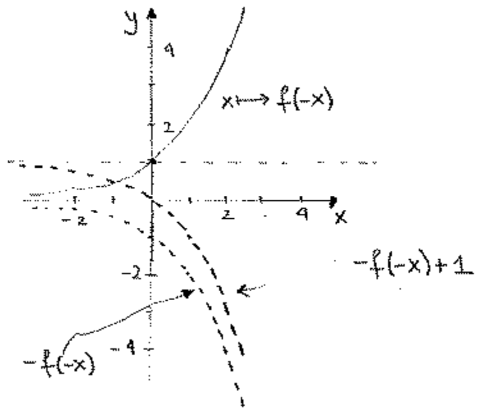


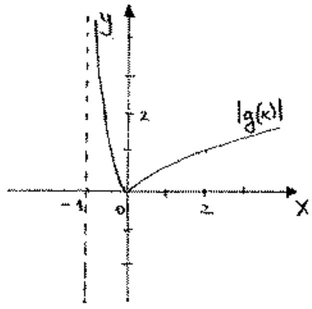
$\mathrm{dom} f = \mathbb{R}$



$\mathrm{dom}\, g = \left]-1,+\infty\right[$

ii)





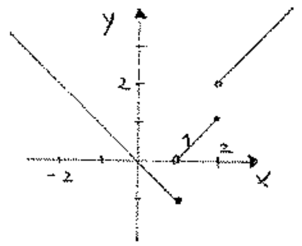
$\mathrm{dom}|g| = \left]-1,+\infty\right[$

iii) $\left(\frac{1}{2}\right)^{x+y} = \left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{y} \quad \forall x,y\in\mathbb{R}$ (proprietà della funzione esponenziale a^x $\forall a>0$)

$g'(x) = \dfrac{1}{x+1}$ $\qquad g''(x) = -\dfrac{1}{(x+1)^2}$ $\qquad$ e quindi $\quad g''(x) = -\left(g'(x)\right)^2 \ \forall x\in \mathrm{dom}\,g.$

iv) Dai grafici di f e di g si vede subito che il nr. delle soluzioni dell'eq. $f(x)=g(x)$ è 1.

4) i) Esempio:



$$f(x) = \begin{cases} -x & \text{se } x\le 1 \\ x-1 & \text{se } 1<x\le 2 \\ x & \text{se } x>2 \end{cases}$$

ii) Esempio: $g(x) = e^x$; allora $g'(x) = e^x > 0 \quad \forall x\in\mathbb{R}$; Esempio: $g(x) = 3x+4 \quad g'(x)=3 \ \forall x\in\mathbb{R}$.

iii) $h'(x) = 3 \ \forall x\in\mathbb{R} \Rightarrow h(x) = 3x+k$ con $k\in\mathbb{R}$ qualsiasi.

Imponendo $\int_1^2 h(x)\,dx = 6$, si ha $\int_1^2 (3x+k)\,dx = 6$. Poiché $\int_1^2 (3x+k)\,dx =$

5) i) $f(x) = \dfrac{x^2}{x-2}$: • $\text{dom} f = \mathbb{R} \setminus \{2\}$

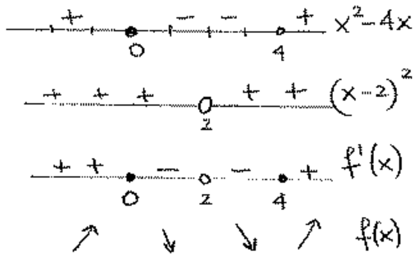
• segno di f : $f(x) > 0$ se $x > 2$
$f(x) \leq 0$ se $x < 2$

• $\lim_{x \to -\infty} f(x) = -\infty$ $\lim_{x \to 2^-} f(x) = -\infty$ $\lim_{x \to 2^+} f(x) = +\infty$ $\lim_{x \to +\infty} f(x) = +\infty$,

• Si ha che $x = 2$ è un asintoto verticale. Si prova inoltre che la retta $y = x + 2$ è un asintoto orizzontale (per $x \to -\infty$, e per $x \to +\infty$).

• f è continua su dom f (rapporto di funzioni continue); inoltre dom f' = dom f e

$$f'(x) = \frac{(x-2)2x - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}.$$



$x = 0$ pt. di max loc. (forte)
$x = 4$ pt di min loc (forte)

$f(0) = 0$

$f(4) = \dfrac{16}{2} = 8$

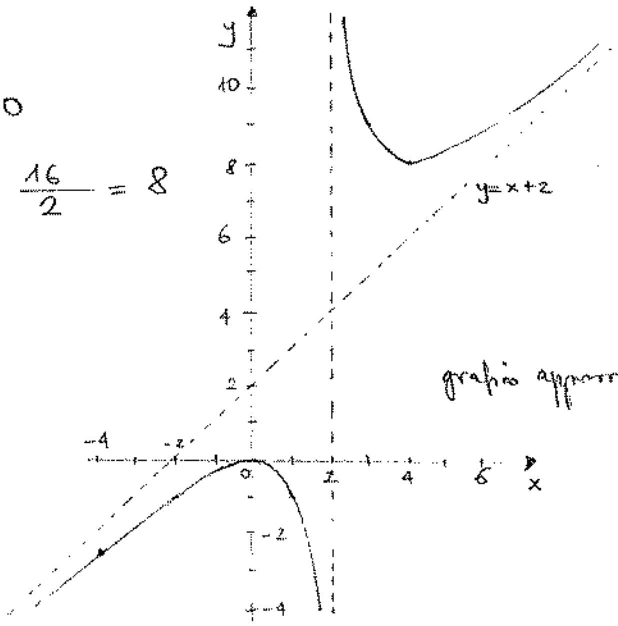


grafico approssimativo di f.

ii) $\min_A f = f(1) = -1$ $\max_A f = f(0) = 0$.

6) $P_{92}^{30,62} = \dfrac{92!}{30!\,62!} = C_{92,30}$.