Verifica Settimanale di ANALISI MATEMATICA (6-10/12/04)

1) i) $\left(e^{x^3+2x}\right)' = e^{x^3+2x}\,(3x^2+2) \qquad \forall x \in \mathbb{R}$

$$\left[\log\left(3x^2+x+\sqrt{x}\right)\right]' = \frac{1}{3x^2+x+\sqrt{x}}\left(6x+1+\frac{1}{2\sqrt{x}}\right) \qquad \forall x>0.$$

$$\left[\frac{x^2}{(x+1)^3}\right]' = \frac{2x(x+1)^{\cancel{3}} - x^2\cdot 3\cancel{(x+1)}^2}{(x+1)^{\cancel{6}\,4}} = \frac{2x(x+1)-3x^2}{(x+1)^4} \qquad \forall x\neq -1.$$

ii) $$\left(\sqrt{x^2+2} + x^2e^{x+1}\right)' = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+2}} + 2xe^{x+1} + x^2e^{x+1}, \qquad \forall x\in\mathbb{R}.$$

$$\left((2x+1)^{-2}\right)' = -2(2x+1)^{-3}\cdot 2 = -4(2x+1)^{-3} \qquad \forall x \neq -\tfrac{1}{2}.$$

$$\left[\frac{(x^4-x^2)2^x}{(3x+1)^2}\right]' = \frac{\left((x^4-x^2)2^x\right)'(3x+1)^{\cancel{2}} - (x^4-x^2)2^x\cdot 2\,\cancel{(3x+1)}\cdot 3}{(3x+1)^{\cancel{4}\,3}}$$

$$= \frac{\left[(4x^3-2x)2^x + (x^4-x^2)2^x\log 2\right](3x+1) - 6\cdot 2^x(x^4-x^2)}{(3x+1)^3} \qquad \forall x\neq -\tfrac{1}{3}. \qquad \square$$

2) i) $$3|x+1| + x^2 > 3x \iff \begin{cases} x+1\geq 0 \\ 3(x+1)+x^2 > 3x \end{cases} \text{ o } \begin{cases} x+1<0 \\ -3(x+1)+x^2>3x \end{cases}$$

$$\iff \begin{cases} x\geq -1 \\ x^2+3>0 \end{cases} \text{ o } \begin{cases} x<-1 \\ x^2-6x-3>0 \end{cases}$$

$$\iff \{x\geq -1 \text{ o } \begin{cases} x<-1 \\ x<3-2\sqrt{2} \text{ o } x>3+2\sqrt{3} \end{cases}$$

$$\iff \{x\geq -1\}\cup\{x<-1\} = \underline{\underline{\mathbb{R}}}.$$

(questo risultato si vede bene anche graficamente: dal grafico si vede che $3|x+1| > 3x \ \forall x\in\mathbb{R}$ e quindi a maggior ragione $3|x+1|+x^2>3x$). $\blacksquare$

$$|x^2-1| + |x| \leq 1 \iff \begin{cases} x^2-1\geq 0 \\ x\geq 0 \\ x^2-1+x\leq 1 \end{cases} \text{ o } \begin{cases} x^2-1<0 \\ x<0 \\ -x^2+1-x\leq 1 \end{cases} \text{ o } \begin{cases} x^2-1\geq 0 \\ x<0 \\ x^2-1-x\leq 1 \end{cases}$$

$$\text{o } \begin{cases} x^2-1<0 \\ x\geq 0 \\ -x^2+1+x\leq 1 \end{cases}$$

$$\iff \begin{cases} x\leq -1 \text{ o } x\geq 1 \\ x\geq 0 \\ x^2+x-2\leq 0 \end{cases} \text{ o } \begin{cases} -1<x<1 \\ x<0 \\ x^2+x\geq 0 \end{cases} \text{ o } \begin{cases} x\leq -1 \text{ o } x\geq 1 \\ x<0 \\ x^2-x-2\leq 0 \end{cases} \text{ o } \begin{cases} -1\leq x\leq 1 \\ x\geq 0 \\ -x^2+x\leq 0 \end{cases}$$

$$\iff \{x=1 \text{ o } x=-1 \text{ o } x=0.$$

$$\iff \underline{\underline{x\in\{-1,0,1\}}}. \qquad \blacksquare$$

$$\begin{array}{lll} -1 & 0 & \\ \hline + + + + & + + + + + & x^2-x+1 \\ \circ - - & \circ + + + + + + + & \log(x+1) \\ \circ - - & \circ + + + + + & \dfrac{x^2-x+1}{\log(x+1)} \end{array} \qquad \begin{array}{lll} -1 & 0 & \\ \hline + + + + & + + + + + & x^2+x+1 \\ \circ - - & \circ + + + + & \log(x+1) \\ \circ - - & \circ + + + + + & \dfrac{x^2+x+1}{\log(x+1)} \end{array}$$

Soluzione: $\underline{\underline{x \in \,]-1, 0[}}$. ■

$$e^{2|x|+1} > \frac{1}{e^x} e^3 \iff e^{2|x|+1} > e^{-x} e^3 \iff e^{2|x|+1} > e^{3-x} \qquad (e^x \nearrow)$$

$$\iff 2|x|+1 > 3-x$$

$$\iff 2|x|+x-2 > 0$$

$$\iff \begin{cases} x \geq 0 \\ 3x-2 > 0 \end{cases} \quad \text{o} \quad \begin{cases} x < 0 \\ -x-2 > 0 \end{cases}$$

$$\iff \underline{\underline{x \in \,]-\infty, -2[\, \cup \,]\tfrac{2}{3}, +\infty[}}. \qquad \square$$

3) i) $\log_2 x + \log_2(x+1) \geq \log_2 3x \iff \begin{cases} x > 0 \\ x+1 > 0 \\ \log_2 x(x+1) \geq \log_2 3x \end{cases}$

$$\iff \begin{cases} x > 0 \\ x(x+1) \geq 3x \end{cases} \iff \begin{cases} x > 0 \\ x^2 - 2x \geq 0 \end{cases} \iff \begin{cases} x > 0 \\ x(x-2) \geq 0 \end{cases} \Rightarrow \underline{\underline{x \in [2, +\infty[}}$$

■

$$\log x^3 - \log|x| < 1 \iff \begin{cases} x > 0 \\ \log \dfrac{x^3}{|x|} < 1 \end{cases} \iff \begin{cases} x > 0 \\ \dfrac{x^3}{|x|} < e \end{cases} \iff \begin{cases} x > 0 \\ \dfrac{x^3}{x} < e \end{cases}$$

$$\iff \begin{cases} x > 0 \\ x^2 < e \end{cases} \Rightarrow \underline{\underline{x \in \,]0, \sqrt{e}[}}. \qquad ■$$

ii) $\left(\frac{1}{2}\right)^{x^2-x} \left(\frac{1}{2}\right)^{2x} \leq 2^{x+1} \iff 2^{-(x^2-x)} \, 2^{-2x} \leq 2^{x+1}$

$$\iff 2^{-x^2+x-2x} \leq 2^{x+1}$$

$$\underset{(2^x \nearrow)}{\iff} -x^2-x \leq x+1 \iff x^2+2x+1 \geq 0$$

$$\Rightarrow \underline{\underline{x \in \mathbb{R}}} \qquad ■$$

$$3^{x^3-x^2} \frac{1}{3^x} > 1 \iff 3^{x^3-x^2} 3^{-x} > 1 \iff 3^{x^3-x^2-x} > 1$$

$$\iff x^3 - x^2 - x > 0 \iff x(x^2-x-1) > 0$$

$$\begin{array}{llll} 1-\sqrt{5} & 0 & 1+\sqrt{5} & \\ \hline - - - - & \bullet + + + + & & x \\ + + \bullet - - & - - & \bullet + + + & x^2-x-1 \end{array}$$

$$\Rightarrow x \in \,]1-\sqrt{5}, 0[\, \cup \,]1+\sqrt{5}, +\infty[$$

4) i) $f(x) = -e^{2x-1}$ $\qquad f'(x) = -e^{2x-1}\cdot 2 = -2e^{2x-1}$

$f'(0) = -\frac{2}{e}$

$$y = -\frac{1}{e} - \frac{2}{e}(x-0) \quad \Rightarrow \quad \underline{\underline{y = -\frac{2}{e}x - \frac{1}{e}}}.$$

ii) $g(x) = \log(e+x^2)$ $\qquad g'(x) = \dfrac{2x}{e+x^2} \qquad g'(0) = 0$

$$\underline{\underline{y = 1}}$$

iii) $h(x) = (3x+1)^3$ $\qquad h'(x) = 3(3x+1)^2\cdot 3 = 9(3x+1)^2$

$h'(-1) = 36$

$$y = -8 + 36(x+1) \quad \Rightarrow \quad \underline{\underline{y = 36x + 28}}$$

$\square$

5) $\boxed{f(x) = x^3 - 2x}$

i) $\text{dom}\, f = \mathbb{R}$ $\qquad$ f è continua su tutto $\mathbb{R}$ (prodotto e somma di funzioni continue) $\qquad$ f è una funzione dispari!

ii) $\lim\limits_{x\to-\infty} f(x) = -\infty \qquad \lim\limits_{x\to+\infty} f(x) = +\infty.$

iii)

$- \; - \; \bullet \; + \; + \; +$ $\qquad x$

$+ \; \bullet \; - \; 0 \; - \; \bullet \; + \; +$ $\qquad (x^2-2)$
$\quad -\sqrt{2} \qquad \sqrt{2}$

$- \; \bullet \; + \; \bullet \; - \; \bullet \; + \; +$ $\qquad x^3-2x$
$-\sqrt{2} \quad 0 \quad \sqrt{2}$

iv) $\text{dom}\, f' = \mathbb{R} \qquad f'(x) = 3x^2 - 2$

$f'(x) = 0 \iff x = -\sqrt{\frac{2}{3}}, \quad x = \sqrt{\frac{2}{3}}$

v) $x = -\sqrt{\frac{2}{3}}$ pt. di max locale
$x = \sqrt{2/3}$ pt. di min locale

$+ \; \bullet \; - \; - \; \bullet \; + \; +$ $\qquad f'$
$-\sqrt{\frac{2}{3}} \quad \sqrt{\frac{2}{3}}$

$\nearrow \quad \searrow \quad \nearrow$ $\qquad f$

$f\left(-\sqrt{\frac{2}{3}}\right) = -\frac{2}{3}\sqrt{\frac{2}{3}} + 2\sqrt{\frac{2}{3}} = \frac{4}{3}\sqrt{\frac{2}{3}} \qquad f\left(\sqrt{\frac{2}{3}}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}}$

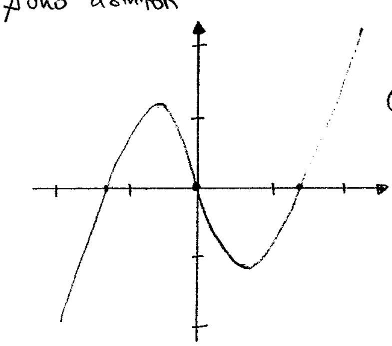
vi) $\text{dom}\, f'' = \mathbb{R} \qquad f''(x) = 6x$

$- \; - \; - \; \bullet \; + \; + \; +$ $\qquad f''$
0

$\frown \qquad \smile$ $\qquad f$

vii) $\nexists$ sono asintoti

viii)



G_f (approssimativo)

$$\boxed{f(x)=\frac{x^2}{x-1}}$$

i) $\mathrm{dom}\, f = \mathbb{R}\setminus\{1\} = \,]-\infty, 1[\, \cup \,]1, +\infty[$

f è continua in tutti i pt. del suo dominio essendo rapporto di funzioni continue

ii) $\lim\limits_{x\to-\infty} f(x) = -\infty$, $\lim\limits_{x\to1^-} f(x) = -\infty$, $\lim\limits_{x\to1^+} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = +\infty$

iii)

	$-\infty$ … 0	0 … 1	1 … $+\infty$
x^2	+ +	+ +	+
$x-1$	− −	−	+
$f(x)$	− − −	−	+

iv) $\mathrm{dom}\, f' = \mathrm{dom}\, f = \mathbb{R}\setminus\{1\}$ $\quad f'(x) = \dfrac{2x(x-1)-x^2}{(x-1)^2} = \dfrac{x^2-2x}{(x-1)^2}$

$f'(x)=0 \iff x(x-2) \iff x=0\,,\ x=2$

v) $x=0$ pt. di max locale

$x=2$ pt. di min locale

$f(0)=0 \quad f(2)=4$

	$-\infty$ … 0	0 … 1	1 … 2	2 … $+\infty$
x^2-2x	+ +	−	−	+ + + +
$(x-1)^2$	+ + +	+	+	+ + +
$f'(x)$	+ +	−	−	+ + + +
f	↗	↘	↘	↗

vi) $\mathrm{dom}\, f'' = \mathrm{dom}\, f'$

$$f''(x) = \frac{(2x-2)(x-1)^{\cancel{2}} - (x^2-2x)\,2\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} =$$

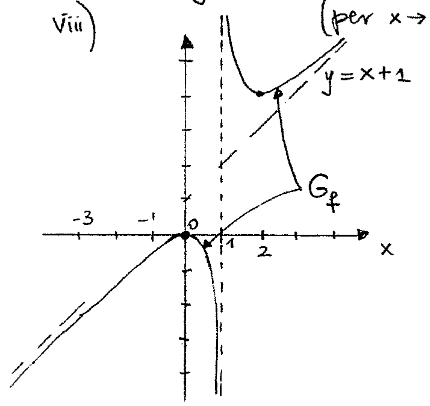
$$= \frac{\cancel{2x^2}\cancel{-4x}+2\cancel{-2x^2}\cancel{+4x}}{(x-1)^3} = \frac{2}{(x-1)^3}$$

	$-\infty$ … 1	1 … $+\infty$
$(x-1)^3$	− − − − − −	+ + + +
$f''(x)=\dfrac{2}{(x-1)^3}$	− − − − − −	+ + + +
f	$\cap$	$\cup$

vii) f ha asintoto verticale $x=1$

f ha asintoto obliquo $y=x+1$ per $x\to+\infty$ (per $x\to-\infty$)

viii)



(grafico approssimativo)

■

$\boxed{f(x) = \frac{x}{x^2-1}}$

i) $\mathrm{dom}\, f = \mathbb{R}\setminus\{-1,1\} = \,]-\infty,-1[\, \cup \,]-1,1[\, \cup \,]1,+\infty[$, (f è una funzione dispari!)

f è continua in tutti i pt. del suo dominio essendo rapporto di funzioni continue

ii) $\lim\limits_{x\to-\infty} f(x) = 0$, $\lim\limits_{x\to-1^-} f(x) = -\infty$, $\lim\limits_{x\to-1^+} f(x) = +\infty$, $\lim\limits_{x\to1^-} f(x) = -\infty$,

$\lim\limits_{x\to1^+} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = 0$.

iii)

```
- - - - 0 + + + + + +       x
+ + o - - o + + + + +       x^2-1
   -1     1
- - - o + • - o + + + +     f(x)
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iv) $\mathrm{dom}\, f' = \mathrm{dom}\, f = \mathbb{R}\setminus\{-1,1\}$ $\quad f'(x) = \frac{(x^2-1) - x(2x)}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2} = -\frac{(x^2+1)}{(x^2-1)^2}$

$f'(x) \neq 0$ sempre (non ci sono pt. critici)

v)

```
-  o  -  o  -     f'(x)
  -1     1
↘     ↘     ↘     f(x)
```

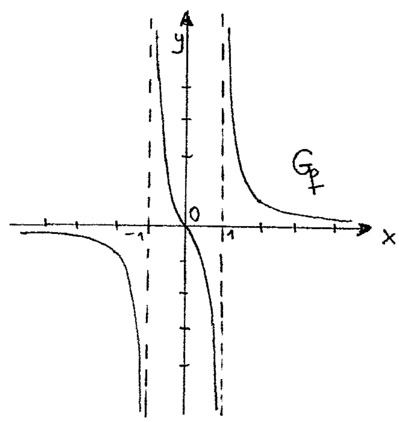
vi) $\mathrm{dom}\, f'' = \mathbb{R}\setminus\{-1,1\}$ $\quad f''(x) = \frac{-2x(x^2-1)^{\cancel{2}} + (x^2+1)\,2\cancel{(x^2-1)}\,2x}{(x^2-1)^{\cancel{4}3}} =$

$= \frac{-2x^3+2x+4x^3+4x}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3}$

```
- - - - - • + + + + +     x
          0
+ + + o - - - o + +       (x^2-1)^3
     -1       1
- - - o + • - o + + +     f''(x)
     -1       1
∩     ∪ ∩      ∪          f(x)
```

vii) f ha 2 asintoti verticali $x=-1$, $x=1$
ha 1 asintoto orizzontale $y=0$ (per $x\to+\infty$
per $x\to-\infty$)

viii)



(grafico approssimativo)

$\boxed{f(x) = (x+1)e^x}$ i) $\operatorname{dom} f = \mathbb{R}$; f è continua in tutti i pt. di $\mathbb{R}$ essendo prodotto di funzioni continue.

ii) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{x+1}{e^{-x}} = 0 \qquad \lim\limits_{x\to+\infty} f(x) = +\infty$.

iii) $- - - \underset{-1}{\bullet} + + +$ $f(x)$ $\qquad (e^x > 0 \ \forall x \in \mathbb{R})$

iv) $\operatorname{dom} f' = \mathbb{R} \qquad f'(x) = e^x + (x+1)e^x = e^x(x+2)$

v) $f'(x) = 0 \iff x = -2$ $\qquad - - - \underset{-2}{\bullet} + + + +$ $f'(x)$

$\searrow \quad \nearrow$ $f(x)$

$x=-2$ pt. di min locale

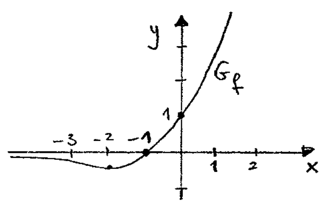
$f(-2) = -e^{-2} = -\frac{1}{e^2}$

vi) $\operatorname{dom} f'' = \mathbb{R} \qquad f''(x) = e^x(x+2) + e^x = e^x(x+3)$

$- - - \underset{-3}{\bullet} + + + +$ $f''(x)$

$\cap \quad \cup$ $f(x)$

vii) f ha un asintoto orizzontale: $y=0$ per $x \to -\infty$.

viii) 

(grafico approssimativo). ■

$\boxed{f(x) = \log(2+x^2)}$ f è una funzione pari!

i) $\operatorname{dom} f = \mathbb{R}$, f è continua in tutti i pt. di $\mathbb{R}$ essendo composizione di funzioni continue.

ii) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = +\infty$.

iii) $f(x) > 0 \ \forall x \in \mathbb{R}$ essendo $2 + x^2 > 1 \ \forall x \in \mathbb{R}$.

iv) $\operatorname{dom} f' = \mathbb{R} \qquad f'(x) = \frac{2x}{2+x^2}$ $\qquad - - - \overset{0}{\bullet} + + + +$ $f'(x)$

$f'(x) = 0 \iff x = 0$ $\qquad \searrow \quad \nearrow$ $f(x)$

v) $x=0$ è pt. di min locale (globale) per f $\qquad f(0) = \log 2$

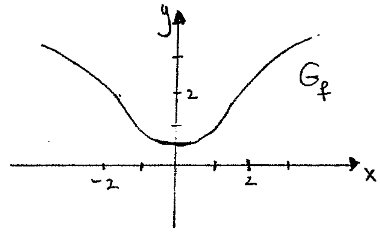
vi) $\operatorname{dom} f'' = \mathbb{R} \qquad f''(x) = \underline{2(2+x^2) - 2x\cdot 2x} = \underline{4-2x^2} = \underline{2(2-x^2)}$.

$$\begin{array}{ccccc} - - - & \bullet & + \; + & \bullet & - - - \\ & -\sqrt{2} & & \sqrt{2} & \end{array} \quad f''(x)$$

$$\cap \qquad \cup \qquad \cap \qquad f(x)$$

vii) non ci sono asintoti ;

viii)



(grafico approssimativo)

$$\boxed{f(x) = \frac{x^2-1}{x^2+1}}$$

i) $\operatorname{dom} f = \mathbb{R}$

f è continua in tutti i pt. di $\mathbb{R}$ essendo rapporto di funzioni continue e la funzione al denominatore non si annulla mai.
f è una funzione pari!

ii) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = 1$.

iii)

$$\begin{array}{ccccc} + \; + & \bullet & - - - & \bullet & + + + \\ & -1 & & 1 & \end{array} \quad f(x)$$

iv) $\operatorname{dom} f' = \mathbb{R}$ $\qquad f'(x) = \dfrac{2x(x^2+1)-(x^2-1)2x}{(x^2+1)^2} = \dfrac{\cancel{2x^3}+2x-\cancel{2x^3}+2x}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$

$f'(x) = 0 \iff x = 0$.

v)

$$\begin{array}{ccc} - - - & \bullet & + + + \\ & 0 & \end{array} \quad f'(x) \qquad \Rightarrow \quad x=0 \text{ è pt. di min. locale per } f$$

$$\searrow \qquad \nearrow \qquad f(x) \qquad\qquad f(0) = -1.$$

vi) $\operatorname{dom} f'' = \mathbb{R}$ $\qquad f''(x) = \dfrac{4(x^2+1)^{\cancel{2}} - 4x \cdot 2\,\cancel{(x^2+1)}\,2x}{(x^2+1)^{\cancel{4}\,3}} = \dfrac{4x^2+4-16x^2}{(x^2+1)^3}$

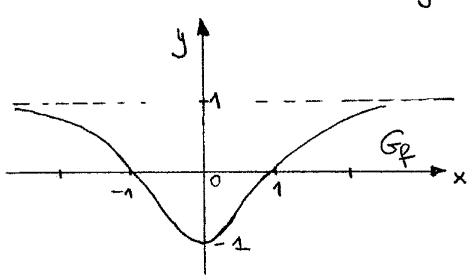
$$= \frac{4(1-3x^2)}{(x^2+1)^3}$$

$$\begin{array}{ccccc} - - & \bullet & + + & \bullet & - - - \\ & -\frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} & \end{array} \quad f''(x)$$

$$\cap \qquad \cup \qquad \cap \qquad f$$

vii) f ha un asintoto orizzontale $y = 1$ $\left(\begin{array}{l}\text{per } x \to +\infty \\ \text{per } x \to -\infty\end{array}\right)$

viii)



(grafico approssimativo)

$\square$