8. Verifica settimanale di ANALISI MATEMATICA, a.a. 2005/06 – 12-16/12/05

1) i) $3+9+27+\dots+2187 = 3^1+3^2+3^3+\dots+3^7 = \underline{\underline{\sum_{i=1}^{7} 3^i}}$;

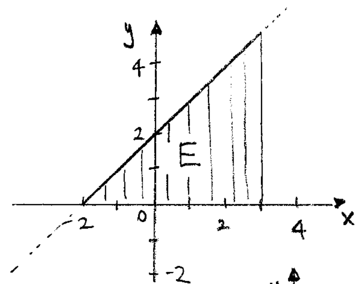
$3-5+7-9+11-13 = \underline{\underline{\sum_{n=1}^{6} (2n+1)(-1)^{n-1}}}$;

ii) $\frac{1}{4}+\frac{1}{8}+\frac{1}{12}+\dots+\frac{1}{52} = \frac{1}{4}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 4}+\dots+\frac{1}{13\cdot 4} = \sum_{m=1}^{13} \frac{1}{4m}$;

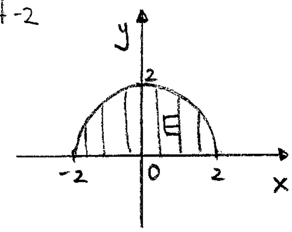
$a^4+a^9+a^{16}+\dots+a^{144} = a^{2^2}+a^{3^2}+a^{4^2}+\dots+a^{12^2} = \sum_{k=2}^{12} a^{k^2}$.

2) i) $\int_{-2}^{3} (x+2)\,dx = \text{area}(E)$

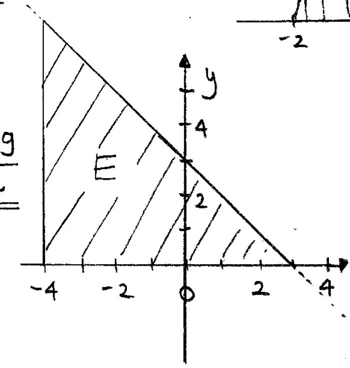
$= \frac{5\cdot 5}{2} = \underline{\underline{\frac{25}{2}}}$;



ii) $\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{\pi 4}{2} = \underline{\underline{2\pi}}$;



iii) $\int_{-4}^{3} (-x+3)\,dx = \frac{7\cdot 7}{2} = \underline{\underline{\frac{49}{2}}}$



3) i) $\int_{-1}^{3} (e^x+x)\,dx = \left[e^x+\frac{x^2}{2}\right]_{-1}^{3} = \left(e^3+\frac{9}{2}\right)-\left(e^{-1}+\frac{1}{2}\right) = \underline{\underline{e^3-\frac{1}{e}+4}}$;

ii) $\int_{0}^{4} \frac{x^2-4}{x+2}\,dx = \int_{0}^{4} (x-2)\,dx = \left[\frac{x^2}{2}-2x\right]_{0}^{4} = (8-8) = \underline{\underline{0}}$;

iii) $\int_{2}^{4} \frac{1}{2x+1}\,dx = \frac{1}{2}\int_{2}^{4} \frac{2}{2x+1}\,dx = \frac{1}{2}\Big[\log|2x+1|\Big]_{2}^{4} = \frac{1}{2}(\log 9-\log 5) = \underline{\underline{\frac{1}{2}\log\frac{9}{5}}}$;

iv) $\displaystyle\int_1^2 \left(e^{2x} + x^{-2}\right) dx = \left[\frac{e^{2x}}{2} - x^{-1}\right]_1^2 = \left(\frac{e^4}{2} - \frac{1}{2}\right) - \left(\frac{e^2}{2} - 1\right) =$

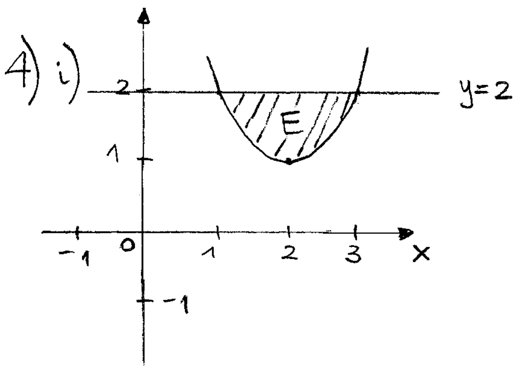
$$= \underline{\underline{\frac{e^4 - e^2}{2} + \frac{1}{2}}} \ ;$$

v) $\displaystyle\int_0^1 \frac{e^{2x}+1}{e^x}\, dx = \int_0^1 \left(e^x + \frac{1}{e^x}\right) dx = \int_0^1 \left(e^x + e^{-x}\right) dx = \left[e^x - e^{-x}\right]_0^1$

$$= \underline{\underline{e - \frac{1}{e}}} \ ;$$

vi) $\displaystyle\int_0^1 (3x^2 + 4x + 2)\, dx = \left[\cancel{3}\frac{x^3}{\cancel{3}} + {}^{2}\cancel{4}\frac{x^2}{\cancel{2}} + 2x\right]_0^1 = 1 + 2 + 2 = \underline{\underline{5}}$

4) i)



$\displaystyle \text{area}(E) = \int_1^3 \left(2 - (x-2)^2 - 1\right) dx =$

$\displaystyle = \int_1^3 \left(2 - x^2 + 4x - 4 - 1\right) dx =$

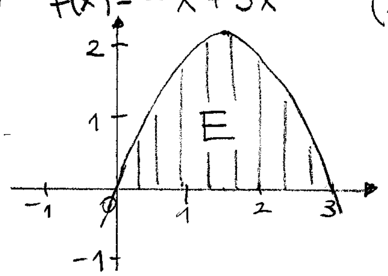
$\displaystyle = \int_1^3 \left(-x^2 + 4x - 3\right) dx$

$\displaystyle = \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$

$= \left(\cancel{-9} + \cancel{18} \cancel{-9}\right) +$

$- \left(-\frac{1}{3} + 2 - 3\right) = \underline{\underline{\frac{4}{3}}}.$

ii) $f(x) = -x^2 + 3x = -\left(x - \frac{3}{2}\right)^2 + \frac{9}{4}$

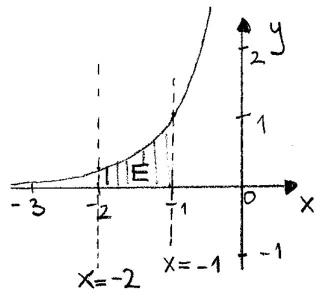


$\displaystyle \text{area}(E) = \int_0^3 f(x)\, dx = \int_0^3 \left(-x^2 + 3x\right) dx$

$\displaystyle = \left[-\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^3 = -9 + \frac{3}{2} \cdot 9$

$$= \underline{\underline{\frac{9}{2}}}.$$

iii)



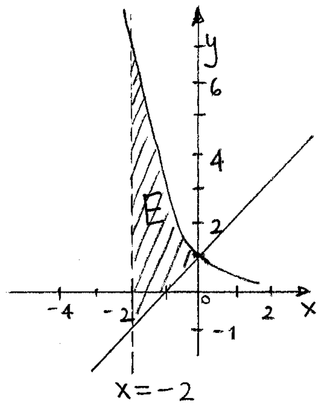
$$\text{area}(E) = \int_{-2}^{-1} \frac{1}{x^4}\,dx = \int_{-2}^{-1} x^{-4}\,dx =$$

$$= \left[\frac{x^{-3}}{-3}\right]_{-2}^{-1} = -\frac{1}{3}\left[\frac{1}{x^3}\right]_{-2}^{-1}$$

$$= -\frac{1}{3}\left(-1+\frac{1}{8}\right) = \underline{\underline{+\frac{7}{24}}}\,;$$

$\square$

iv)



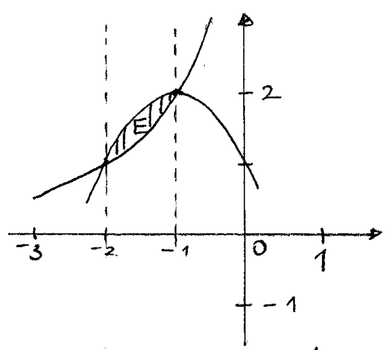
$$\text{area}(E) = \int_{-2}^{0} \left(e^{-x} - x - 1\right) dx =$$

$$= \left[-e^{-x} - \frac{x^2}{2} - x\right]_{-2}^{0} =$$

$$= (-1) - \left(-e^2 - \cancel{2} + \cancel{2}\right) = \underline{\underline{e^2 - 1}}\,;$$

$\square$

v) $f(x) = -x^2 - 2x + 1 = -(x+1)^2 + 2$



$$f(x) = g(x) \iff$$

$$-x^2 - 2x + 1 = -\frac{2}{x}$$

$$\iff \quad -x^3 - 2x^2 + x = -2$$

$$x = -1 \qquad x = -2$$

$$\text{area}(E) = \int_{-2}^{-1} \left(f(x) - g(x)\right) dx = \int_{-2}^{-1} \left(-x^2 - 2x + 1 + \frac{2}{x}\right) dx$$

$$= \left[-\frac{x^3}{3} - x^2 + x + 2\log|x|\right]_{-2}^{-1} =$$

$$= \left(\frac{1}{3} - 1 - 1 + \cancel{2\log 1}\right) - \left(\frac{8}{3} - 4 - 2 + 2\log 2\right) =$$

$$= \underline{\underline{\frac{5}{3} - \log 4}}\,.$$

$\blacksquare$

5) a) $f(x) = (x^2-4)^3$

funzione pari!

i) dom $f = \mathbb{R}$; f è continua in tutti i pt. di $\mathbb{R}$.

ii) $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$

iii) segno di f

x		-2		0		2	
x^2-4	$+$	$\bullet$	$-$	$-$	$-$	$\bullet$	$+\ +$

ha lo stesso segno $f(x)$

iv) dom $f' = \mathbb{R}$,

$$f'(x) = 3(x^2-4)^2 \cdot 2x$$

$\Rightarrow$ $x=0$, $x=-2$, $x=2$ sono pt. critici di f

v)

x	-4	-2		0		2	4
$(x^2-4)^2$	$+$	$\bullet$	$+$	$+$	$+$	$\bullet$	$+$
x	$-$	$-$	$-$	$\bullet$	$+$	$+$	$+$
$f'(x)$	$-$	$\bullet$	$-$	$\bullet$	$+$	$\bullet$	$+$
$f(x)$	$\searrow$						$\nearrow$

$x=0$ è un pt. di min loc. (stretto) di f ; e $f(0) = -64$.

vi) dom $f'' = \mathbb{R}$

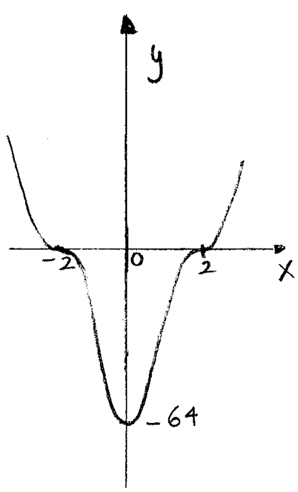
$$f''(x) = 6(x^2-4)4x^2 + 3(x^2-4)^2 \cdot 2$$
$$= 6(x^2-4)\left[4x^2 + x^2 - 4\right] = 6(x^2-4)(5x^2-4)$$

$f''(x) = 0 \iff x = -2$, $x = 2$, $x = -\frac{2}{\sqrt{5}}$ $x = \frac{2}{\sqrt{5}}$

x	-4	-2	$-\frac{2}{\sqrt5}$	0	$\frac{2}{\sqrt5}$	2	4
x^2-4	$+$	$\bullet$	$-$	$-$	$-$	$\bullet$	$+\ +$
$5x^2-4$	$+$	$+$	$\bullet$	$-$	$\bullet$	$+$	$+$
$f''(x)$	$+$	$\bullet$ $-$	$\bullet$ $+$		$\bullet$ $-$	$\bullet$	$+\ +$
f	$\cup$	$\cap$	$\cup$		$\cap$		$\cup$

vii) grafico approssimativo di f $\square$

(ovviamente ho scelto due unità di misura diverse sull'asse x e sull'asse y)



b) $f(x) = \dfrac{1}{1-x^2}$

funzione pari!

i) $\operatorname{dom} f = \mathbb{R}\setminus\{-1,1\} =]-\infty,-1[\,\cup\,]-1,1[\,\cup\,]1,+\infty[$;

f è continua in tutti i pt. del dominio.

ii) $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0 \Rightarrow y=0$ è asintoto orizzontale per $x\to+\infty$, (per $x\to-\infty$)

$\lim_{x\to-1^-} f(x) = -\infty$, $\lim_{x\to-1^+} f(x) = +\infty \Rightarrow x=-1$ è asintoto verticale;

$\lim_{x\to1^-} f(x) = +\infty$, $\lim_{x\to1^+} f(x) = -\infty \Rightarrow x=1$ è asintoto verticale.

iii) segno di f

$- \quad \circ_{-1} \quad + \quad \circ_{1} \quad -$ $\qquad 1-x^2$

iv) $\operatorname{dom} f' = \operatorname{dom} f$

$f'(x) = \dfrac{+2x}{(1-x^2)^2}$ $\qquad x=0$ pt. critico di f

v)

$- \;_{-2}\; - \;\bullet_{0}\; + \;_{2}\; +$ $\qquad 2x$

$+ \;\circ_{-1}\; + \;\circ_{1}\; +$ $\qquad (1-x^2)^2$

$- \; - \;\circ_{-1}\; - \;\bullet_{0}\; + \;\circ_{1}\; + \; +$ $\qquad f'(x)$

$\searrow \quad \searrow \; \nearrow \quad \nearrow$ $\qquad f(x)$

$x=0$ è pt. di min. loc (stretto) di f e $f(0)=1$.

vi) $\operatorname{dom} f'' = \operatorname{dom} f$

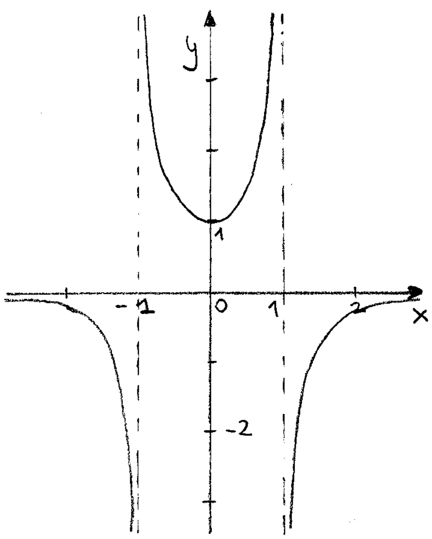
$$f''(x) = \frac{2(1-x^2)^{\cancel{2}} - 2x\cdot 2\cancel{(1-x^2)}(-2x)}{(1-x^2)^{\cancel{4}3}}$$

$$= \frac{2-2x^2+8x^2}{(1-x^2)^3} = \frac{6x^2+2}{(1-x^2)^3}$$

$- \; - \;\circ_{-1}\; + \;\circ_{1}\; - \; -$ $\qquad$ segno di f''

$\cap \qquad \cup \qquad \cap$ $\qquad f$

vii) grafico approssimativo di f



$\square$

c) $f(x) = \dfrac{2x^2}{x^2-1}$

funzione pari!

i) $\mathrm{dom}\, f = \mathbb{R}\setminus\{-1,1\} = \,]-\infty,-1[\,\cup\,]-1,1[\,\cup\,]1,+\infty[$;

f è continua in tutti i pt. del dominio.

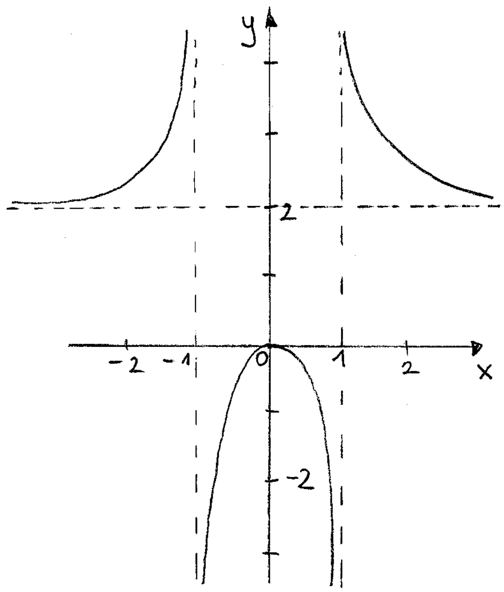
ii) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = 2 \Rightarrow y=2$ è asintoto orizzontale per $x\to-\infty$; (per $x\to+\infty$)

$\lim\limits_{x\to-1^-} f(x) = +\infty \quad \lim\limits_{x\to-1^+} f(x) = -\infty \Rightarrow x=-1$ è asintoto verticale;

$\lim\limits_{x\to1^-} f(x) = -\infty \quad \lim\limits_{x\to1^+} f(x) = +\infty \Rightarrow x=1$ è asintoto verticale.

iii) segno di f

	$x<-1$	-1	$-1<x<0$	0	$0<x<1$	1	$x>1$
x^2	+		+	•	+		+
x^2-1	+	o	−		−	o	+
$f(x)$	+	o	−	•	−	o	+



iv) $\mathrm{dom}\, f' = \mathrm{dom}\, f$

$$f'(x) = \frac{4x(x^2-1) - 2x^2\cdot 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$\Rightarrow \quad x=0$ è pt. critico di f

v)

	$x<-1$	-1	$-1<x<0$	0	$0<x<1$	1	$x>1$
$-4x$	+		+	•	−		−
$(x^2-1)^2$	+	o	+		+	o	+
$f'(x)$	+	o	+	•	−	o	−
$f(x)$	↗		↗		↘		↘

$\Rightarrow \; x=0$ è pt. di max loc. (stretto) di f e $f(0)=0$

OSS. $f(x) = 2 + \dfrac{2}{x^2-1} =$

$= 2 - \dfrac{2}{1-x^2}$;

quindi si poteva ottenere il grafico di questa funzione considerando la funzione in b), moltiplicata per per -2 e traslata poi di 2 lungo l'asse delle y !!

vi) $\mathrm{dom}\, f'' = \mathrm{dom}\, f$

$$f''(x) = \frac{-4(x^2-1)^{\cancel{2}} + 4x\cdot 2\cancel{(x^2-1)}\cdot 2x}{(x^2-1)^{\cancel{4}\,3}} =$$

$$= \frac{12x^2+4}{(x^2-1)^3}$$

	$x<-1$	-1	$-1<x<1$	1	$x>1$
segno di $f''(x)$	+	o	−	o	+
	∪		∩		∪

vii) grafico approssimato di f $\square$

d) $f(x) = x^2 e^{-2x}$

i) dom $f = \mathbb{R}$; f è continua in tutti i pt. di $\mathbb{R}$;

ii) $\lim\limits_{x\to-\infty} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = 0$

$\Rightarrow$ $y=0$ è asintoto orizzontale per $x\to+\infty$.

iii) segno di f

$$\begin{array}{cccccccccc} + & & + & & \bullet & + & & + & & + \\ & -1 & & & 0 & & 1 & & 2 & \end{array}$$

iv) dom $f' = \mathbb{R}$ $\quad f'(x) = 2x e^{-2x} - 2x^2 e^{-2x} = 2e^{-2x}(x-x^2)$

$$= 2e^{-2x}\, x\,(1-x)$$

$\Rightarrow$ $x=0$, $x=1$ sono pt. critici di f

v)

$$\begin{array}{ccccccccl} - & & - & \bullet & + & & + & & x \\ & -1 & & 0 & & 1 & & 2 & \end{array}$$

$$\begin{array}{ccccccccl} + & & + & & + & \bullet & - & & 1-x \\ & -1 & & 0 & & 1 & & 2 & \end{array}$$

$$\begin{array}{ccccccl} - & & - & \bullet\ + & \bullet & - & f'(x) \\ & -1 & & 0 & 1 & & \\ & \searrow & & \nearrow & & \searrow & f(x) \end{array}$$

$x=0$ pt. di min. locale (stretto) di f ; $f(0)=0$

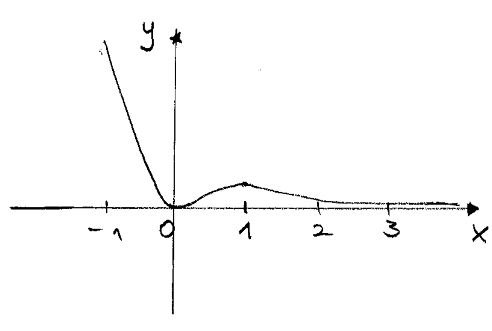
$x=1$ pt. di max locale (stretto) di f ; $f(1) = \frac{1}{e^2}$

vi) dom $f'' = \mathbb{R}$ $\quad f''(x) = -4e^{-2x}(x-x^2) + 2e^{-2x}(1-2x)$

$$= 2e^{-2x}\,(2x^2-4x+1)$$

$$\begin{array}{cccccccccl} + & + & & \bullet & - & - & \bullet & + & + & 2x^2-4x+1 \\ & & 0 & \frac{2-\sqrt{2}}{2} & & & \frac{2+\sqrt{2}}{2} & & & \\ & \cup & & & \cap & & & \cup & & f(x) \end{array}$$

vii) grafico approssimativo di f



e) $f(x) = x \log(x^2)$

i) $\operatorname{dom} f = \mathbb{R} \setminus \{0\} =]-\infty, 0[\cup]0, +\infty[$;

f è continua in tutti i punti del dominio ;

ii) $\lim\limits_{x \to -\infty} f(x) = -\infty$, $\lim\limits_{x \to +\infty} f(x) = +\infty$

$\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = 0$

iii) segno di f

$- \; - \; - \; \bullet \; + \; +$ x ($-2 \;\; -1 \;\; 0 \;\; 1 \;\; 2$)

$+ \; + \; \bullet \; - \; \circ \; - \; \bullet \; + \; +$ $\log(x^2)$ ($-1 \;\; 0 \;\; 1$)

$- \; \bullet \; + \; \circ \; - \; \bullet \; +$ $f(x)$ ($1 \;\; 0 \;\; 1$)

iv) $\operatorname{dom} f' = \operatorname{dom} f$

$f'(x) = \log(x^2) + \cancel{x} \cdot \frac{1}{\cancel{x^2}} \cdot 2\cancel{x}$

$= \log(x^2) + 2$ $x = -\frac{1}{e}$ $x = \frac{1}{e}$ pt. critici di f

v) $+ \; + \; + \; \bullet \; - \; \circ \; - \; \bullet \; + \; + \; +$ $f'(x)$ ($-1 \;\; -\frac{1}{e} \;\; \frac{1}{e} \;\; 1$)

$\nearrow \;\; \searrow \; \searrow \;\; \nearrow$ $f(x)$

$\Rightarrow x = -\frac{1}{e}$ pt. di max. loc. (stretto) di f

$f\left(-\frac{1}{e}\right) = -\frac{1}{e} \log\left(\frac{1}{e^2}\right) = \frac{2}{e}$

$\Rightarrow x = \frac{1}{e}$ pt. di min. loc. (stretto) di f

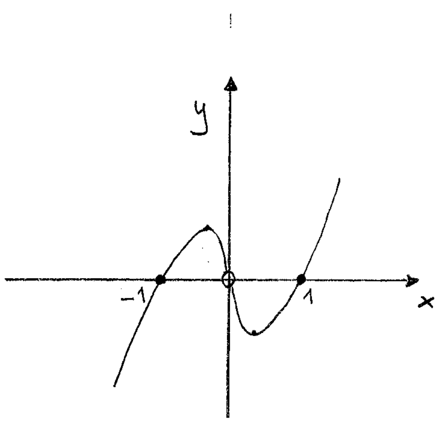
$f\left(\frac{1}{e}\right) = \frac{1}{e} \log\left(\frac{1}{e^2}\right) = -\frac{2}{e}$.

vi) $\operatorname{dom} f'' = \operatorname{dom} f$

$f''(x) = \frac{2x}{x^2} = \frac{2}{x}$

$- \; - \; \circ \; + \; + \; +$ $f''(x)$ ($-1 \;\; 0 \;\; 1$)

$\cap \qquad \cup$ $f(x)$



vii) grafico approssimativo di f