Università degli Studi di Trento — Facoltà di Scienze Cognitive
CdL in Interfacce e Tecnologie della Comunicazione
CdL in Scienze e Tecniche di Psicologia Cognitiva
Esame scritto di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2008–2009 — Rovereto, 9 febbraio 2009

Compito (A)

1) i) E :
$$x^2+2x+4(y^2-4y+4)=3$$
$$x^2+2x+1+4(y^2-4y+4)=4$$
$$(x+1)^2+4(y-2)^2=4 \iff \frac{(x+1)^2}{4}+(y-2)^2=1$$

$\underline{C=(-1,2)}$

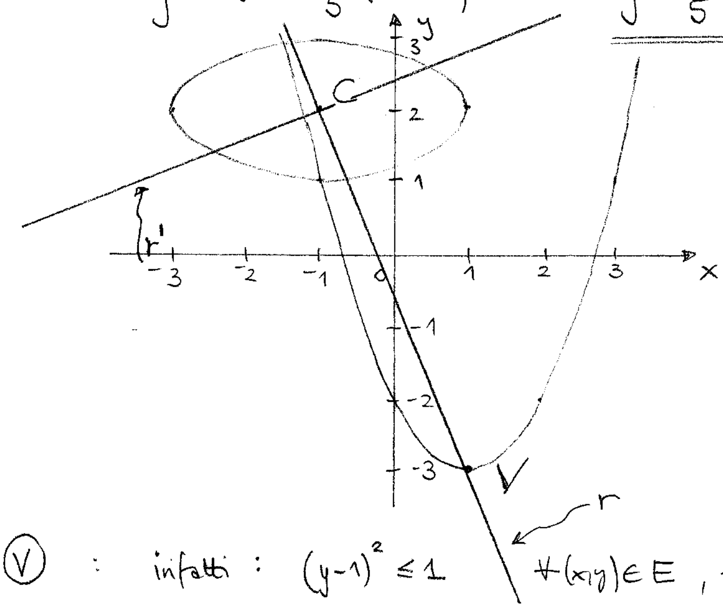
(semiassi $a=2$, $b=1$)

parabola : $y-x^2+2x+2=0 \iff y=x^2-2x-2$
$$y=(x-1)^2-3 \qquad \underline{V=(1,-3)}$$

r : $\frac{y-2}{-3-2}=\frac{x+1}{1+1} \iff y-2=-\frac{5}{2}(x+1)$
$$\underline{y=-\frac{5}{2}x-\frac{1}{2}}.$$ □

ii) r' : $y=2+\frac{2}{5}(x+1) \implies \underline{y=\frac{2}{5}x+\frac{12}{5}}$. □

iii)



□

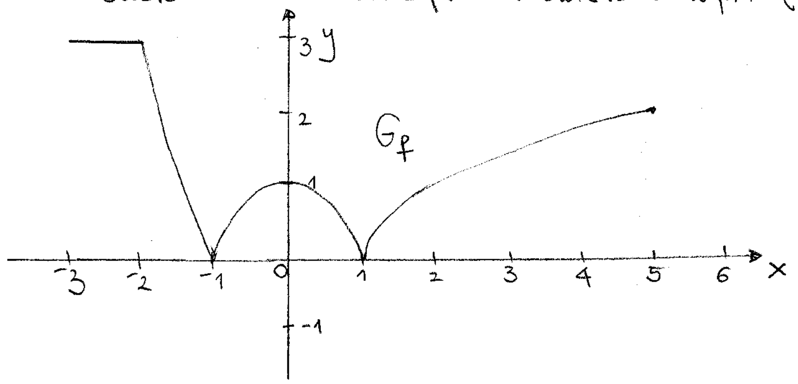
iv)
a) (V) : infatti : $(y-1)^2 \le 1$ $\forall (x,y)\in E$, e mi ha $-1\le (y-2) \le 1$
e quindi $1\le y\le 3$ $\forall (x,y)\in E$; "basta" comunque una motivazione "grafica".

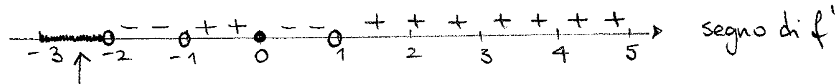
b) (F) : basta osservare che $(1,2)\in E$.

c) V : infatti $(1,2) \in E$.

d) V : basta inserire nell'eq. dell'ellisse E il pt. $(-1,1)$.

2) i)



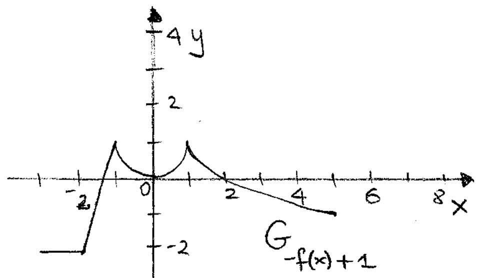


$f'(x) \equiv 0$ su $[-3,-2[$

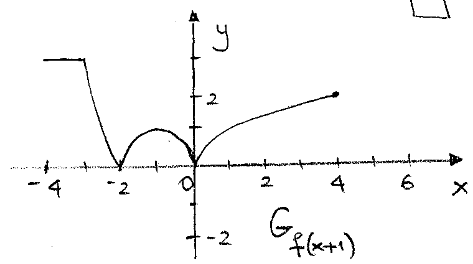
ii) $x=-1$, $x=1$ pt. di minimo di f su $[-1,2]$; $\min\limits_{[-1,2]} f = \underline{\underline{0}}$

$x=0$, $x=2$ pt. di massimo di f su $[-1,2]$; $\max\limits_{[-1,2]} f = \underline{\underline{1}}$.

iii)



$\mathrm{dom}\,(-f(x)+1) = [-3,5]$



$\mathrm{dom}\, f(x+1) = [-4,4]$.

iv) $\displaystyle\int_1^5 f(x)\,dx = \int_1^5 \sqrt{x-1}\,dx = \int_1^5 (x-1)^{1/2}\,dx = \left[\frac{(x-1)^{3/2}}{3/2}\right]_1^5 = \frac{2}{3}\left[8-0\right] = \underline{\underline{\frac{16}{3}}}$

3) i) $2^{|x-1|}\cdot 2^x < 4 \iff 2^{|x-1|+x} < 2^2 \underset{2^x \uparrow}{\iff} |x-1|+x<2$

$\iff \begin{cases} x \geq 1 \\ x-1+x<2 \end{cases}$ oppure $\begin{cases} x<1 \\ -\cancel{x}+1+\cancel{x}<2 \end{cases}$

$\iff \begin{cases} x \geq 1 \\ 2x<3 \end{cases}$ opp. $\begin{cases} x<1 \\ \mathbb{R} \end{cases}$ $\Rightarrow$ $\underline{\underline{S = \left]-\infty, \frac{3}{2}\right[}}$;

$\log_2(3-x^2) \leq 1 \iff \begin{cases} 3-x^2>0 \\ 3-x^2 \leq 2 \end{cases} \iff \begin{matrix} x^2<3 \\ x^2 \geq 1 \end{matrix}$

$\Rightarrow$ $\underline{\underline{S = \left]-\sqrt{3},-1\right] \cup \left[1,\sqrt{3}\right[}}$;

$$\frac{1}{||x-1|-2|} > 0 \iff ||x-1|-2| \neq 0 \iff |x-1|-2 \neq 0$$

$$\iff |x-1| \neq 2 \iff x-1 \neq 2 \text{ opp. } x-1 \neq -2$$

$$\Rightarrow S = \mathbb{R} \setminus \{-1, 3\}.$$

ii) $$\sum_{n=1}^{8} \int_n^{n+1} \frac{1}{x^2}\,dx = \int_1^2 \frac{1}{x^2}\,dx + \int_2^3 \frac{1}{x^2}\,dx + \dots + \int_8^9 \frac{1}{x^2}\,dx = \int_1^9 \frac{1}{x^2}\,dx = \int_1^9 x^{-2}\,dx =$$

$$= \left[\frac{x^{-1}}{-1}\right]_1^9 = \left[-\frac{1}{x}\right]_1^9 = -\frac{1}{9} + 1 = \frac{8}{9}.$$

4) $f : [0,1] \to \mathbb{R}$ $\quad f(x) = \dfrac{1}{e^x + 1}$

i) V : $e^x \geq 1 \ \forall x \in [0,1]$ e quindi $\dfrac{1}{e^x+1} \leq \dfrac{1}{2} = f(0)$.

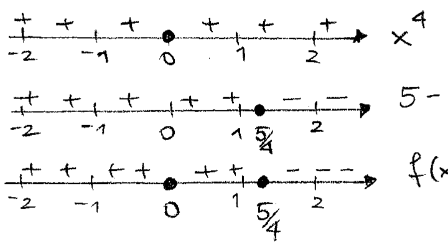
ii) F : infatti $\displaystyle\int_0^1 \frac{1}{f(x)}\,dx = \int_0^1 (e^x+1)\,dx = \left[e^x + x\right]_0^1 = e + \cancel{1} - \cancel{1} = e > 1.$

iii) V : per il teorema di Weierstrass.

iv) V : $\displaystyle\lim_{x\to+\infty} \frac{f'(x)}{f(x)} = \lim_{x\to+\infty} \frac{-e^x}{(e^x+1)^{\cancel{2}}} \cdot \cancel{(e^x+1)} = -1$

5) i) $f(x) = 5x^4 - 4x^5$

- dom $f = \mathbb{R} =]-\infty, +\infty[$; f è continua
- segno di $f(x) = x^4(5-4x)$



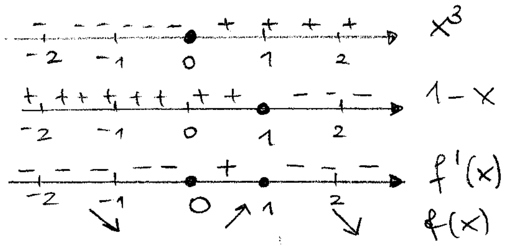
- $\displaystyle\lim_{x\to-\infty} f(x) = +\infty \qquad \lim_{x\to+\infty} f(x) = -\infty$

- dom $f' = \mathbb{R}$ $\qquad f'(x) = 20x^3 - 20x^4$

$$= 20x^3(1-x)$$

$f'(x) = 0 \iff x = 0,\ x = 1$ pt. critici di f

monotonia di f



Abbiamo dunque che $x=0$ è pt. di min. loc. (stretto) per f, $\underline{\underline{f(0)=0}}$

$x=1$ è pt. di max. loc. (stretto) per f $\underline{\underline{f(1)=1}}$;

• dom $f'' = \mathbb{R}$ $\quad f''(x) = 60x^2 - 80x^3 = 20x^2(3-4x)$

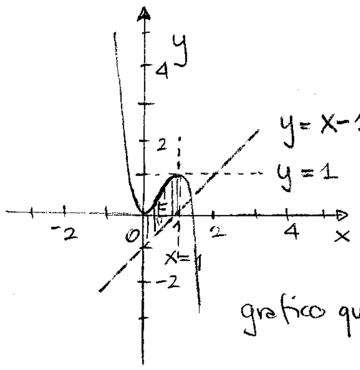
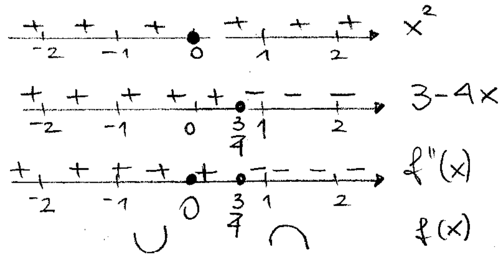


grafico qualitativo di f



$x = \frac{3}{4}$ pt. di flesso

ii) $\underline{\underline{y=1}}$ □

iii) se $k<0$ oppure $k>1$ $\exists!$ una soluzione dell'eq. $f(x)=k$;

$k=0$ oppure $k=1$ $\exists$ sono 2 soluzioni "

$0<k<1$ $\exists$ sono 3 soluzioni " . □

iv) area $E = \int_0^1 \left(f(x) - (x-1)\right)dx$

$$= \int_0^1 \left(5x^4 - 4x^5 - x + 1\right)dx =$$

$$= \left[\cancel{5}\frac{x^5}{\cancel{5}} - \overset{2}{\cancel{4}}\frac{x^6}{\cancel{6}_3} - \frac{x^2}{2} + x\right]_0^1 = 1 - \frac{2}{3} - \frac{1}{2} + 1 = 2 - \frac{7}{6} = \underline{\underline{\frac{5}{6}}}$$

■

6) $C_{n,2} = 10 \iff \frac{n!}{2!\,(n-2)!} = 10 \iff \frac{n(n-1)\cancel{(n-2)!}}{2\cdot\cancel{(n-2)!}} = 10$

$\iff n^2 - n = 20 \iff n^2 - n - 20 = 0 \iff n_{1/2} = \frac{1 \pm \sqrt{1+80}}{2}$

$\iff n_{1/2} = \frac{1 \pm 9}{2} = \begin{cases} 5 \\ -4 \text{ non accettabile.} \end{cases}$

Soluzione : $\underline{\underline{n=5}}$.

■

Compito (B)

1) i) E : $\quad x^2 - 2x + 4(y^2 + 4y + 4) = 3$

$$x^2 - 2x + 1 \; + 4(y^2 + 4y + 4) = 4$$

$$(x-1)^2 + 4(y+2)^2 = 4 \iff \left(\frac{x-1}{4}\right)^2 + (y+2)^2 = 1$$

$$\underline{\underline{C = (1,-2)}}$$

(semiassi $a = 2$, $b = 1$)

parabola : $\quad y + x^2 + 2x - 2 = 0 \iff y = -x^2 - 2x + 2$

$$y = -(x^2 + 2x) + 2$$

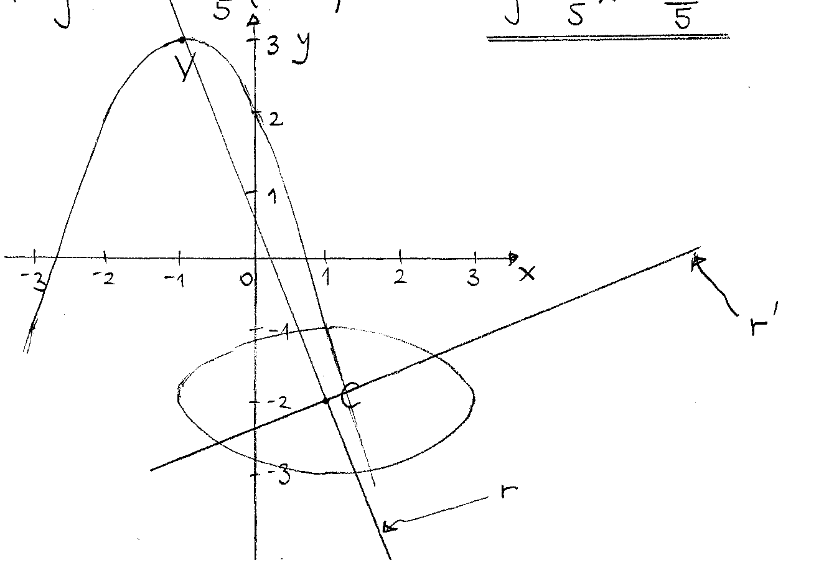
$$y = -(x+1)^2 + 3 \qquad \underline{\underline{V = (-1,3)}}$$

r : $\quad \frac{y+2}{3+2} = \frac{x-1}{-1-1} \iff y + 2 = -\frac{5}{2}(x-1)$

$$\underline{\underline{y = -\frac{5}{2}x + \frac{1}{2}}}$$

ii) r' : $\quad y = -2 + \frac{2}{5}(x-1) \Rightarrow \underline{\underline{y = \frac{2}{5}x - \frac{12}{5}}}$.

iii)



iv) a) (V) : infatti : $(y+2)^2 \leq 1 \;\; \forall (x,y) \in E$, e si ha $-1 \leq y+2 \leq 1$ e quindi $-3 \leq y \leq -1 \;\; \forall (x,y) \in E$; "basta" comunque una motivazione "grafica".

b) (F) : basta osservare che $(-1,-2) \in E$.

c) (V) : infatti $(-1,-2) \in E$ (oppure $(3,-2) \in E$).

d) (F) : $(1,-2)$ è il centro dell'ellisse .

2) i) $2^{|x-1|}\cdot 2^{x} \geq 8 \iff 2^{|x-1|+x} \geq 2^{3} \underset{2^x \uparrow}{\iff} |x-1|+x \geq 3$

$\iff \begin{cases} x \geq 1 \\ x-1+x \geq 3 \end{cases}$ oppure $\begin{cases} x<1 \\ \cancel{-x}+1+\cancel{x} \geq 3 \end{cases}$

$\iff \begin{cases} x \geq 1 \\ 2x \geq 4 \end{cases}$ oppure $\begin{cases} x<1 \\ \emptyset \end{cases} \Rightarrow \underline{\underline{S=[2,+\infty[}}\,;$

$\log_3(4-x^2)<1 \iff \begin{cases} 4-x^2>0 \\ 4-x^2<3 \end{cases} \iff \begin{cases} x^2<4 \\ x^2>1 \end{cases}$

$\Rightarrow \underline{\underline{S=\,]-2,-1[\,\cup\,]1,2[}}\,;$

$\dfrac{1}{||x-1|-3|}>0 \iff ||x-1|-3| \neq 0 \iff |x-1|-3 \neq 0$

$\iff |x-1| \neq 3 \iff x-1 \neq 3$ opp. $x-1 \neq -3$

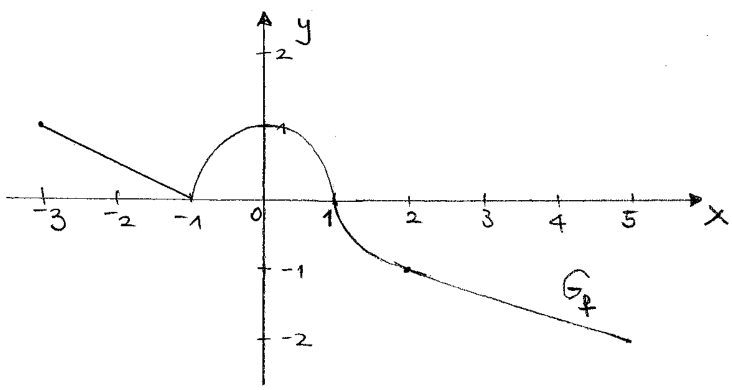
$\Rightarrow \underline{\underline{S=\mathbb{R}\setminus\{-2,4\}}}.$ □

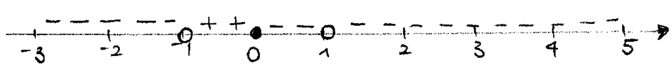
ii) $\displaystyle\sum_{n=1}^{8}\int_n^{n+1}\frac{1}{x^3}\,dx=\int_1^2\frac{1}{x^3}dx+\int_2^3\frac{1}{x^3}dx+\dots+\int_8^9\frac{1}{x^3}dx=\int_1^9\frac{1}{x^3}dx=\int_1^9 x^{-3}dx$

$\displaystyle=\left[\frac{x^{-2}}{-2}\right]_1^9=\left[-\frac{1}{2\cdot x^2}\right]_1^9=-\frac{1}{162}+\frac{1}{2}=\frac{80}{162}=\underline{\underline{\frac{40}{81}}}$ ■

3) vedi Compito A, Es. 4. ■

4) i)



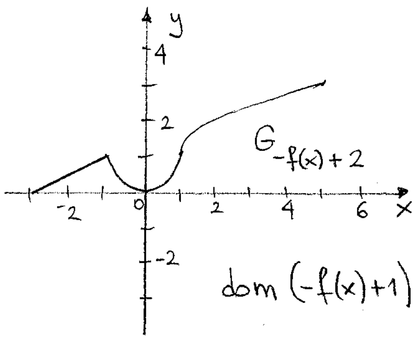


□

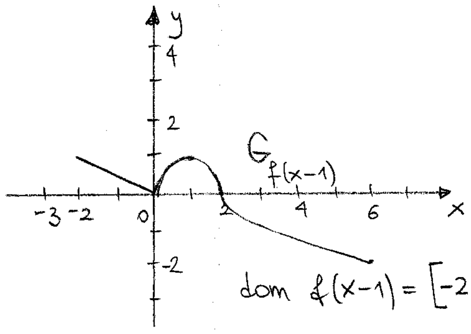
ii) $x=-1$, $x=1$ pt. di minimo di f in $[-3,1]$; $\quad \min\limits_{[-3,1]} f = \underline{\underline{0}}$

$x=-3$, $x=0$ pt. di massimo di f in $[-3,1]$; $\quad \max\limits_{[-3,1]} f = \underline{\underline{1}}.$ □

iii)



$\text{dom}\,(-f(x)+1) = [-3,5]$

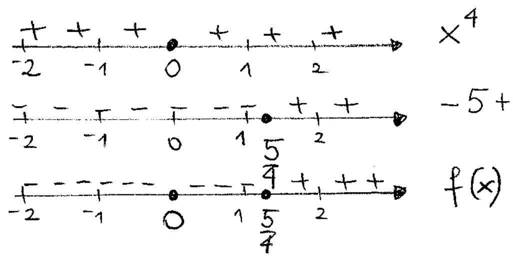


$\text{dom}\, f(x-1) = [-2,6]$

iv) $$\int_2^5 f(x)\,dx = -\int_2^5 \sqrt{x-1}\,dx = -\int_2^5 (x-1)^{\frac{1}{2}}\,dx = -\left[\frac{(x-1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^5$$
$$= -\frac{2}{3}\left[8-1\right] = \underline{\underline{-\frac{14}{3}}}$$

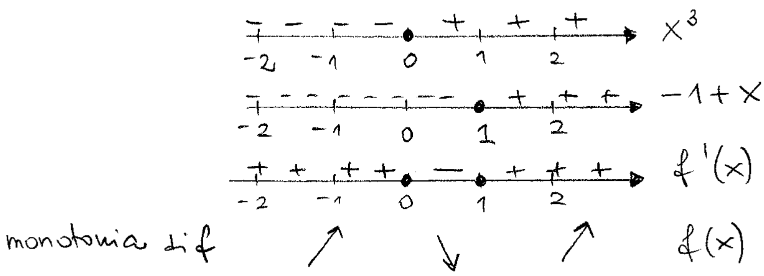
5) i) $f(x) = -5x^4 + 4x^5$

- $\text{dom} f = \mathbb{R} =]-\infty,+\infty[$; f è continua
- segno di $f(x) = x^4(-5+4x)$



- $\lim\limits_{x\to-\infty} f(x) = \underline{\underline{-\infty}}$ $\quad \lim\limits_{x\to+\infty} f(x) = \underline{\underline{+\infty}}$
- $\text{dom} f' = \mathbb{R}$ $\quad f'(x) = -20x^3 + 20x^4$
$$= 20x^3(-1+x)$$

$f'(x) = 0 \iff \underline{\underline{x=0,\ x=1 \text{ pt. critici di } f}}$



Abbiamo dunque che $x=0$ pt. di max. loc. stretto per f; $\underline{\underline{f(0)=0}}$

$x=1$ pt. di min. loc. stretto per f; $\underline{\underline{f(1)=-1}}$;

• $\text{dom } f'' = \mathbb{R}$ $\quad f''(x) = -60x^2 + 80x^3$

$$= 20x^2(-3+4x)$$

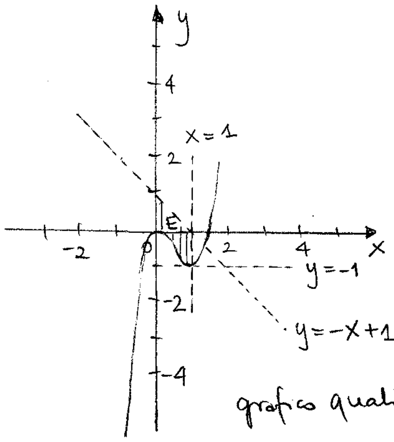
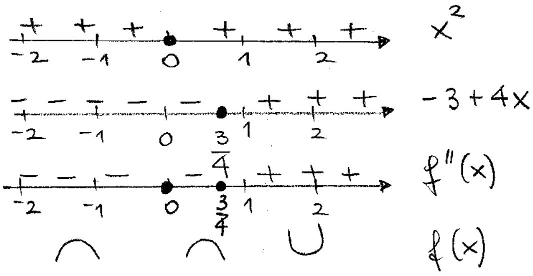


grafico qualitativo di f



$x = \frac{3}{4}$ pt. di flesso

ii) $\underline{y = -1}$

iii) se $k < -1$ oppure $k > 0$ $\quad \exists!$ una soluzione dell'eq. $f(x) = k$;

$k = -1$ oppure $k = 0$ $\quad$ sono 2 soluzioni "

$-1 < k < 0$ $\quad$ sono 3 soluzioni " .

iv) $$\text{area } E = \int_0^1 \left(-x+1 - f(x)\right)dx =$$

$$= \int_0^1 \left(-x+1+5x^4-4x^5\right)dx$$

$$= \left[-\frac{x^2}{2} + x + \cancel{5}\frac{x^5}{\cancel{5}} - \overset{2}{\cancel{4}}\frac{x^6}{\cancel{6}_3}\right]_0^1 = -\frac{1}{2} + 1 + 1 - \frac{2}{3} = \underline{\underline{\frac{5}{6}}}$$

6) $$D_{n,2} = 12 \iff \frac{n!}{(n-2)!} = 12 \iff \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 12$$

$$\iff n^2 - n = 12 \iff n^2 - n - 12 = 0 \iff n_{1/2} = \frac{1 \pm \sqrt{1+48}}{2}$$

$$\iff n_{1/2} = \frac{1 \pm 7}{2} = \begin{cases} 4 \\ -3 \text{ non accettabile.} \end{cases}$$

Soluzione : $\underline{\underline{n = 4.}}$