Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva –
CdL in Interfacce e Tecnologie della Comunicazione
Esame scritto di Analisi Matematica (con elementi di Algebra)
a.a. 2008–2009 – Rovereto, 11 settembre 2009

Fila (A)

(1) i) $A=\{x\in\mathbb{R}: |x^2+3x|<4\} = \,]-4,1[$.

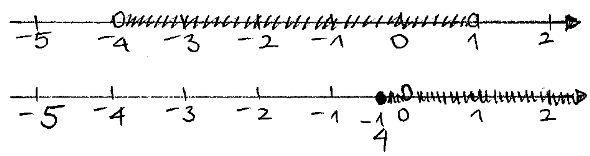
$|x^2+3x|<4 \iff -4<x^2+3x<4$

$\iff \begin{cases} x^2+3x+4>0 \\ x^2+3x-4<0 \end{cases} \iff \begin{cases} x\in\mathbb{R} \\ -4<x<1 \end{cases}$

$B=\left\{x\in\mathbb{R}: \dfrac{x^3+4x+1}{x^2}\geq x\right\} = \left[-\tfrac14,+\infty\right[\setminus\{0\}$.

$\dfrac{x^3+4x+1}{x^2}\geq x \iff \dfrac{\cancel{x^3}+4x+1-\cancel{x^3}}{x^2}\geq 0$

$\iff \dfrac{4x+1}{x^2}\geq 0 \iff x\in\left[-\tfrac14,+\infty\right[\setminus\{0\}$

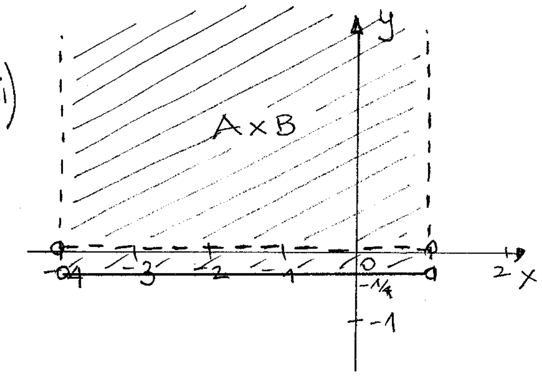


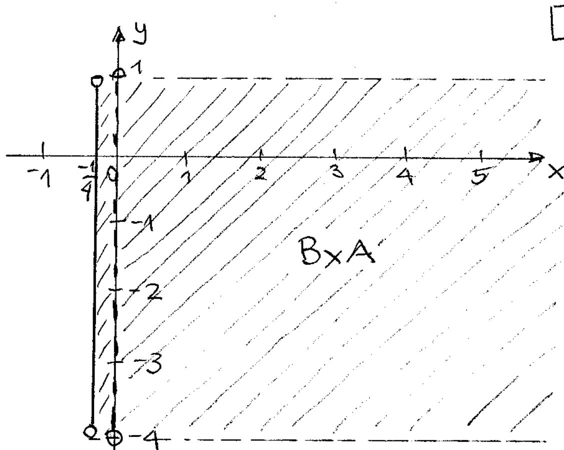
//// = A

//// = B □

ii) $A\cup B = \,]-4,+\infty[$ $\quad A\cap B=\left[-\tfrac14,1\right[\setminus\{0\}$ $\quad A\setminus B = \,\left]-4,-\tfrac14\right[\cup\{0\}$. □

iii)





$y=k$ con $k\in\left[-\tfrac14,+\infty\right[\setminus\{0\}$

sono le rette orizzontali che intersecano l'insieme $A\times B$. □

iv) a) (V) : il rettangolo $[-3,-1]\times[1,4]$ è contenuto nell'insieme $A\times B$;

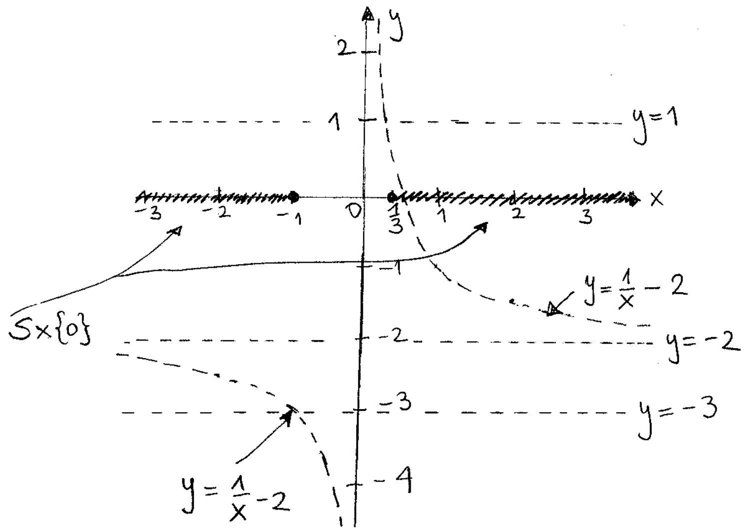
b) (F) : è erronea la scrittura; sarebbe corretta la scrittura $[-3,-1]\times[1,4] \in \mathcal{P}(A\times B)$;

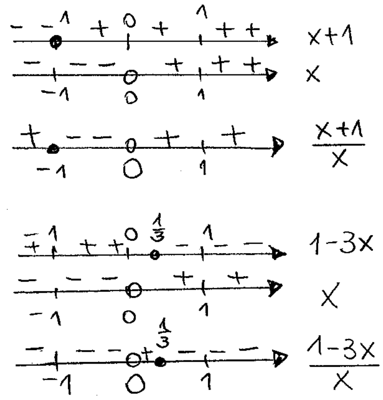
c) (F) poiché $0 \notin B$;

d) (V) $[-3,-1]\times[1,4]$ è un sottoinsieme di $A\times B$.

(2) • $-3 \leq \frac{1}{x}-2 \leq 1 \iff -1 \leq \frac{1}{x} \leq 3 \iff \begin{cases} \frac{1}{x}+1 \geq 0 \\ \frac{1}{x}-3 \leq 0 \end{cases}$

$\iff \begin{cases} \frac{x+1}{x} \geq 0 \\ \frac{1-3x}{x} \leq 0 \end{cases}$





$$S = \left]-\infty,-1\right] \cup \left[\tfrac{1}{3},+\infty\right[.$$

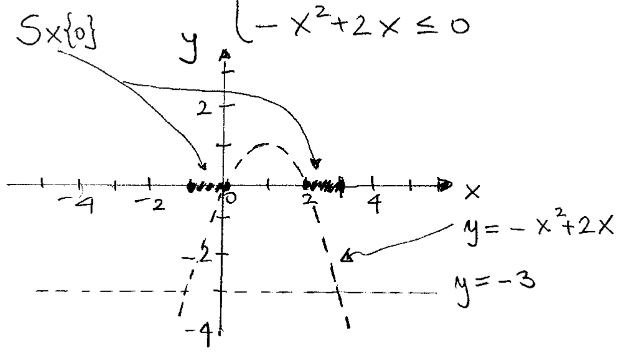
• $-3 \leq -x^2+2x \leq 0 \iff \begin{cases} -x^2+2x+3 \geq 0 \\ -x^2+2x \leq 0 \end{cases}$

Poiché $-x^2+2x+3=0 \iff x_{1/2} = \frac{-2\pm\sqrt{4+12}}{-2} = \frac{-2\pm4}{-2} = \begin{cases} 3 \\ -1 \end{cases}$

$-x^2+2x=0 \iff x_{1/2} = \begin{cases} 0 \\ 2 \end{cases}$

si ha $\begin{cases} -x^2+2x+3 \geq 0 \\ -x^2+2x \leq 0 \end{cases} \iff \begin{cases} x\in[-1,3] \\ x\in\left]-\infty,0\right]\cup\left[2,+\infty\right[\end{cases}$

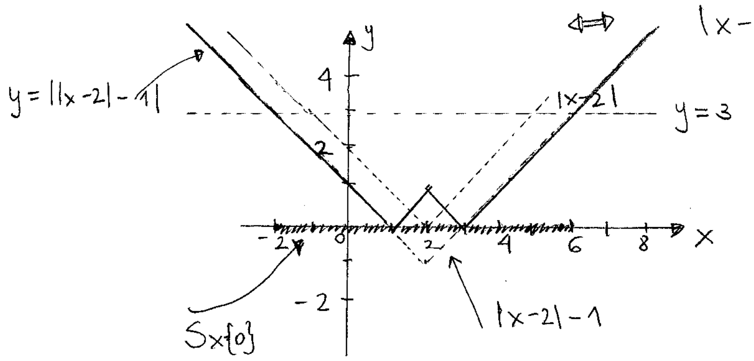
$\Rightarrow$ $S = [-1,0] \cup [2,3]$.



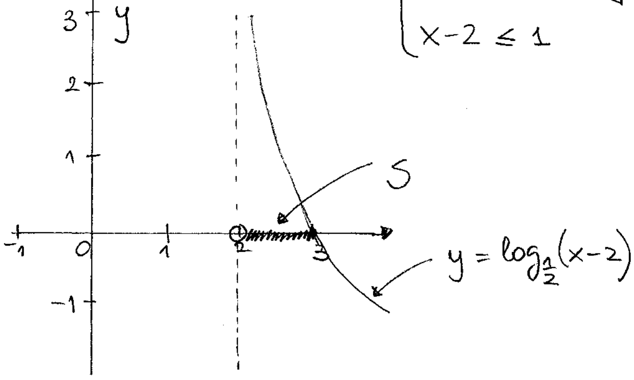
- $||x-2|-1| \le 3 \iff -3 \le |x-2|-1 \le 3 \iff \underset{\uparrow \text{sempre}}{-2 \le} |x-2| \le 4$

  $\iff |x-2| \le 4 \iff -4 \le x-2 \le 4 \Rightarrow$

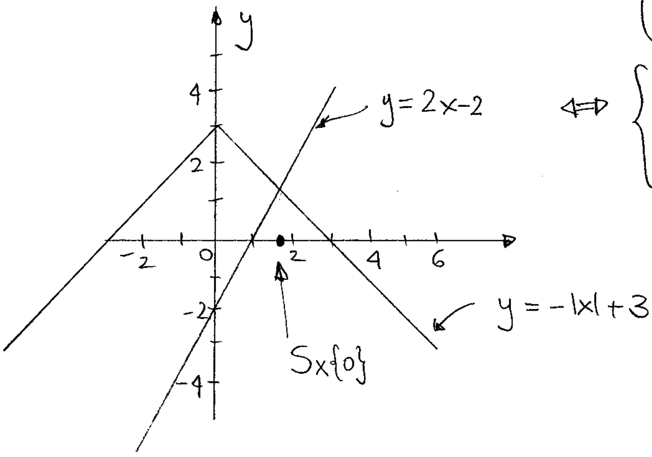
  $S = [-2, 6]$.



- $\log_{\frac{1}{2}}(x-2) \ge 0 \iff \begin{cases} x-2 > 0 \\ x-2 \le 1 \end{cases} \iff \begin{cases} x > 2 \\ x \le 3 \end{cases} \Rightarrow S =]2, 3]$.



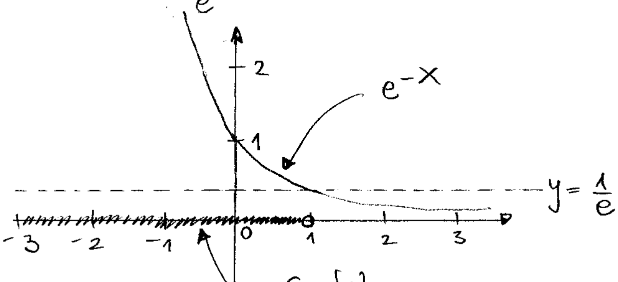
- $2x - 2 = -|x| + 3 \iff \begin{cases} x \ge 0 \\ 2x-2 = -x+3 \end{cases}$ o $\begin{cases} x < 0 \\ 2x-2 = x+3 \end{cases}$



$\iff \begin{cases} x \ge 0 \\ 3x = 5 \end{cases}$ o $\begin{cases} x < 0 \\ x = 5 \end{cases}$

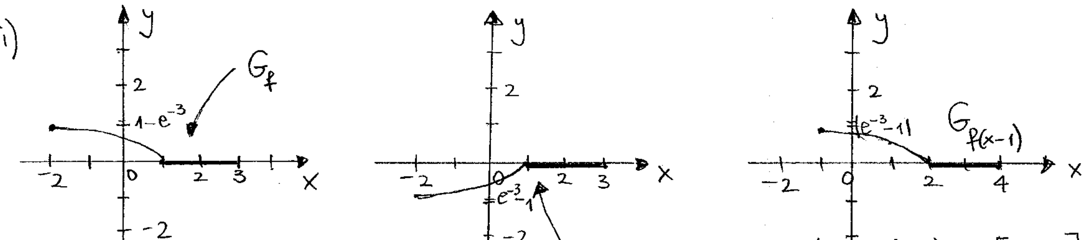
$\Rightarrow S = \left\{\frac{5}{3}\right\}$

- $e^{-x} > \frac{1}{e} \iff e^{-x} > e^{-1} \iff x < 1$ $S =]-\infty, 1[$



3 $f(x) = \begin{cases} |e^{x-1} - 1| & \text{se } -2 \le x \le 1 \\ 0 & \text{se } 1 < x \le 3 \end{cases}$

i) i)



dom $f(x-1) = [-1, 4]$

ii) $F(x) = \int_{-2}^{x} f(t)\,dt$ è strettamente crescente su $[-2, 1]$, poiché

$f(x) > 0$ su $[-2, 1[$ e $f(x) \equiv 0$ per $x \in [1, 3]$.

iii) $F(x) = \int_{-2}^{x} (1 - e^{t-1})\,dt$ se $x \in [-2, 1]$ risulta

$$= \Big[t - e^{t-1}\Big]_{-2}^{x} = \left(x - e^{x-1}\right) - \left(-2 - e^{-3}\right)$$

$$= x - e^{x-1} + 2 + \frac{1}{e^3}$$

Poiché $f \equiv 0$ su $[1, 3]$

$$F(x) = \begin{cases} x - e^{x-1} + 2 + \frac{1}{e^3} & \text{se } x \in [-2, 1] \\ 2 + \frac{1}{e^3} & \text{se } x \in]1, 3]. \end{cases}$$

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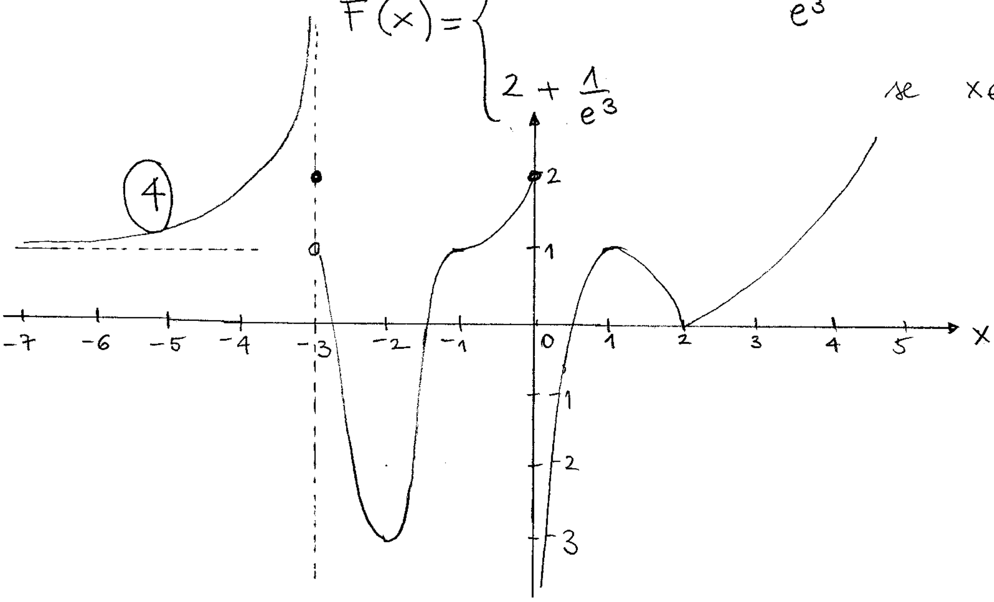


i) $\lim_{x \to -\infty} f(x) = 1$; $\lim_{x \to -3^-} f(x) = +\infty$; $\lim_{x \to -3^+} f(x) = 1$; $\lim_{x \to 0^-} f(x) = 2$; $\lim_{x \to 0^+} f(x) = -\infty$

$\lim_{x \to +\infty} f(x) = +\infty$.

ii) $x=-2$; $x=-1$; $x=1$.

iii) $\int_{-3}^{-1} f(x)\,dx < 0$; $\int_{-1}^{0} f(x)\,dx > 0$.

iv) $\min_{[-3,0]} f = f(-1) = -3$, $\max_{[-3,0]} f = f(-3) = f(0) = 2$.

No; perché f non è continua in $[-3,0]$.

(5) i) $f(x) = e^{-x^2}\cdot e^{x} - 1$ • dom $f = \mathbb{R} =]-\infty,+\infty[$

$= e^{-x^2+x} - 1$ • segno di f

$$\begin{array}{c} - - - \;\; + + \;\; - - - - \\ \overset{}{-1 \quad 0 \quad 1 \quad 2} \end{array} \quad f$$

$-x^2+x = 0 \iff x=0,\ x=1$

$\left[e^{-x^2+x} - 1 > 0 \iff e^{-x^2+x} > 1 \iff -x^2+x > 0 \right]$

• $\lim_{x\to-\infty} f(x) = -1$ $\lim_{x\to+\infty} f(x) = -1$ $y=-1$ è asint. orizz. per f (per $x\to-\infty$, per $x\to+\infty$)

• dom $f' = \mathbb{R}$ $f'(x) = (-2x+1)e^{-x^2+x}$ $\left(e^{-x^2+x} > 0 \ \forall x\in\mathbb{R}\right)$

$f'(x) = 0 \iff -2x+1 = 0 \iff x = \frac{1}{2}$ pt. critico di f

$$\begin{array}{c} + + + \quad\; - - - \\ -1 \quad 0 \quad 1 \quad 2 \\ \nearrow \qquad \searrow \end{array} \quad \begin{array}{c} f' \\ \\ f \end{array}$$

• $x = \frac{1}{2}$ pt. di max. loc. stretto $f\left(\frac{1}{2}\right) = e^{\frac{1}{4}} - 1 = \sqrt[4]{e} - 1$.

• dom $f'' = \mathbb{R}$ $f''(x) = (-2x+1)^2 e^{-x^2+x} - 2e^{-x^2+x}$

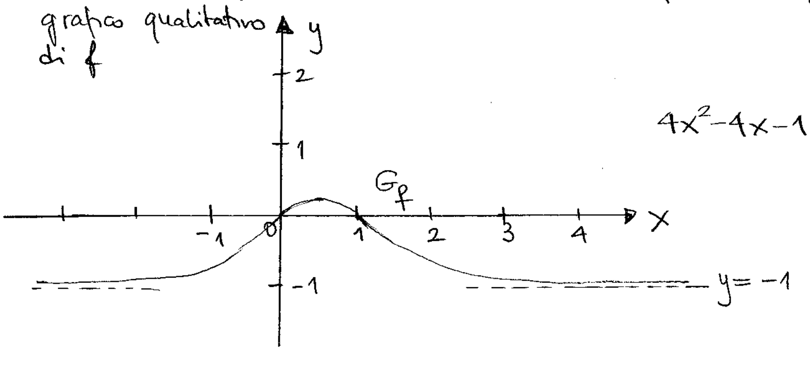
$= (4x^2 - 4x - 1)e^{-x^2+x}$

$4x^2-4x-1 = 0 \iff x_{1/2} = \frac{4\pm\sqrt{16+16}}{8} =$

$= \frac{4\pm4\sqrt{2}}{8} = \frac{1}{2} \pm \frac{\sqrt{2}}{2}$

$$\begin{array}{c} + + \quad - - \quad + + \\ -1 \quad 0 \quad 1 \quad 2 \\ \tfrac{1}{2}-\tfrac{\sqrt2}{2} \qquad \tfrac{1}{2}+\tfrac{\sqrt2}{2} \\ \cup \qquad \cap \qquad \cup \end{array} \quad \begin{array}{c} f'' \\ \\ \\ f \end{array}$$

grafico qualitativo di f



ii) $P = (0, f(0)) = (0,0)$ $y = 0 + f'(0)(x-0) = 1\cdot x$ $\Rightarrow$ $y = x$ è l'eq. della retta tg. al grafico di f in P.

iii) $\int_0^2 (1-2x)f(x)\,dx = \int_0^2 (1-2x)e^{-x^2+x}\,dx - \int_0^2 (1-2x)\,dx = \left[e^{-x^2+x}\right]_0^2 - \left[x - x^2\right]_0^2$

$= (e^{-2} - 1) - (-2) = \underline{\underline{\frac{1}{e^2} + 1}}$.

(6) $C_{13,4}\cdot C_{6,1}\cdot C_{6,1}\cdot C_{6,1}\cdot C_{6,1} = \underline{\underline{\frac{13!}{4!\,9!}\cdot 6^4}}$.