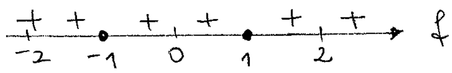
Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Interfacce e Tecnologie della Comunicazione
CdL in Scienze e Tecniche di Psicologia Cognitiva
Verifica Settimanale di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a. a. 2008 – 2009 – Rovereto, 15 – 19 dicembre 2008

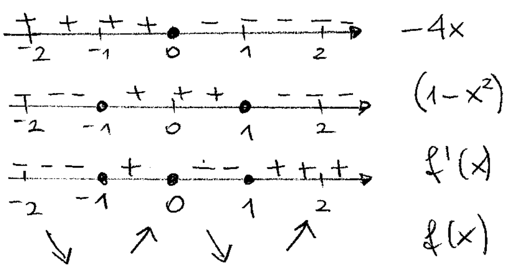
1) $f(x) = (1-x^2)^2$

OSS: f è una funzione pari! (Il grafico di f è simmetrico rispetto all'asse y)

- $\operatorname{dom} f = \mathbb{R} =]-\infty, +\infty[$
- segno di f:



- $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$.
- f è continua in $\mathbb{R}$, poiché composizione di funzioni continue su $\mathbb{R}$.
- $\operatorname{dom} f' = \mathbb{R}$ $\quad f'(x) = 2(1-x^2)(-2x) = -4x(1-x^2)$



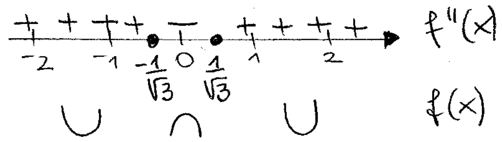
$f'(x) = 0 \iff x=-1,\ x=0,\ x=+1$ pt. critici di f

$x=-1$ pt. di min. loc. stretto $\quad f(-1) = 0$

$x=0$ pt. di max loc. stretto $\quad f(0) = 1$

$x=1$ pt. di min. loc. stretto $\quad f(1) = 0$.

- $\operatorname{dom} f'' = \mathbb{R}$ $\quad f''(x) = -4 + 12x^2$ $\quad f''(x) = 0 \iff x^2 = \frac{1}{3} \Rightarrow x_{1/2} = \pm\frac{1}{\sqrt{3}}$



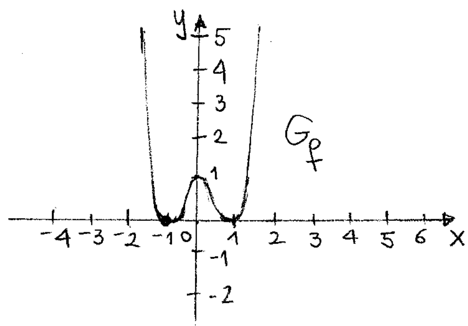


grafico qualitativo di f

□

$g(x) = \log\left(\frac{x}{x-1}\right)$

- $\operatorname{dom} g = \left\{x \in \mathbb{R} : \frac{x}{x-1} > 0\right\} =]-\infty, 0[\cup]1, +\infty[$

$- - - \bullet + + + \quad x$
$- - - - \circ + + \quad x-1$
$+ + + \bullet - \circ + + \quad \frac{x}{x-1}$
$-2 \;\; -1 \;\; 0 \;\; 1 \;\; 2$

- segno di g: $- - - \; 0 \; 1 \; + +$

g non è definita su $[0,1]$.

$\frac{x}{x-1} > 1 \iff \frac{x}{x-1} - 1 > 0 \iff \frac{\cancel{x} - \cancel{x} + 1}{x-1} > 0 \iff x > 1$

- $\lim_{x \to -\infty} \log\left(\frac{x}{x-1}\right) = 0$ $\quad$ ($\frac{x}{x-1} \searrow 1$)

$\lim_{x \to 0^-} \log\left(\frac{x}{x-1}\right) = -\infty$

infinitesimo positivo

$\lim_{x \to 1^+} \log\left(\frac{x}{x-1}\right) = +\infty$ $\quad$ ($\frac{x}{x-1} \to +\infty$)

$\lim_{x \to +\infty} \log\left(\frac{x}{x-1}\right) = 0.$ $\quad$ ($\frac{x}{x-1} \nearrow 1$)

$y=0$ a. o. per g
(per $x \to -\infty$, per $x \to +\infty$)

- g è continua nel suo dominio, poiché composizione di funz. continue.
- $\operatorname{dom} g' =]-\infty, 0[\cup]1, +\infty[$

$$g'(x) = \frac{1}{\frac{x}{x-1}} \cdot \left(\frac{x}{x-1}\right)' = \frac{x-1}{x} \cdot \frac{(x-1) - x \cdot 1}{(x-1)^2} = \frac{\cancel{x-1}}{x} \cdot \frac{-1}{(x-1)^{\cancel{2}}} = \frac{-1}{x(x-1)}$$

manca

$+ + + \; 0 \; 1 \; - - - - \quad -x$
$- - - \; 0 \; 1 \; + + + \quad (x-1)$
$- - - \; 0 \; 1 \; - - - \quad g'(x)$
$-2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4$
$\searrow \qquad \searrow \qquad g(x)$

- $\operatorname{dom} g'' =]-\infty, 0[\cup]1, +\infty[$ $\qquad g''(x) = \frac{(x-1) + x}{x^2(x-1)^2} = \frac{2x-1}{x^2(x-1)^2}$

manca

$- - - \; 0 \; 1 \; + + + \quad g''(x)$
$-2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4$
$\cap \qquad \cup \qquad g(x)$

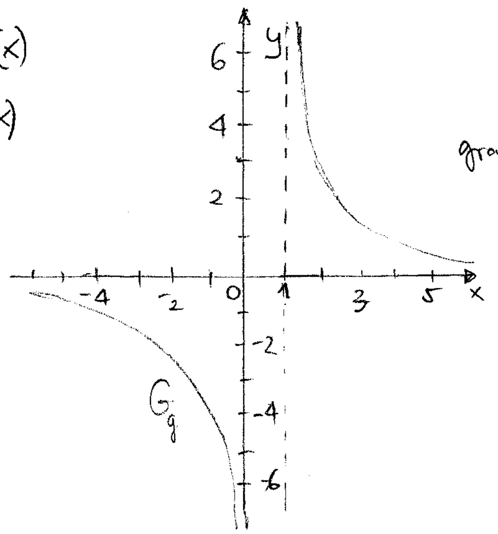


grafico qualitativo di g

$\square$

$$h(x) = \frac{x^2}{x^2-2}$$

- $\text{dom}\, h = \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\} = \underline{]-\infty, -\sqrt{2}[\cup]-\sqrt{2}, \sqrt{2}[\cup]\sqrt{2}, +\infty[}$

- segno di h :

	-2		$-\sqrt{2}$	-1		0		1	$\sqrt{2}$	2	
x^2	+	+	+	+	+	•	+	+	+	+	+
x^2-2	+	+	0	−	−	−	−	−	0	+	+
h	+	+	0	−	−	•	−	−	0	+	+

- $\lim\limits_{x \to -\infty} h(x) = 1$ $\quad y = 1$ a.o. per h (per $x \to -\infty$)

$\lim\limits_{x \to -\sqrt{2}^-} h(x) = +\infty$ $\quad \left(\dfrac{x^2 \to 2}{x^2-2 \text{ infinitesimo positivo}} \longrightarrow +\infty \right)$ $\quad x = -\sqrt{2}$ a.v. per h

$\lim\limits_{x \to -\sqrt{2}^+} h(x) = -\infty$ $\quad \left(\dfrac{x^2 \to 2}{x^2-2 \text{ infinitesimo negativo}} \longrightarrow -\infty \right)$ $\quad$ "

$\lim\limits_{x \to \sqrt{2}^-} h(x) = -\infty$ $\quad \left(\dfrac{x^2 \to 2}{x^2-2 \text{ infinitesimo negativo}} \longrightarrow -\infty \right)$ $\quad x = \sqrt{2}$ a.v. per h

$\lim\limits_{x \to \sqrt{2}^+} h(x) = +\infty$ $\quad \left(\dfrac{x^2 \to 2}{x^2-2 \text{ infinitesimo positivo}} \longrightarrow +\infty \right)$ $\quad$ "

$\lim\limits_{x \to +\infty} h(x) = 1$ $\quad y = 1$ a.o. per h (per $x \to +\infty$).

- h è continua nel suo dominio, poiché rapporto di funzioni continue.
- $\text{dom}\, h' = \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$ $\quad h'(x) = \dfrac{2x(x^2-2) - x^2 \cdot 2x}{(x^2-2)^2} = \dfrac{-4x}{(x^2-2)^2}$

	-2		$-\sqrt{2}$	-1		0		1	$\sqrt{2}$	2	
$-4x$	+	+	+	+	+	•	−	−	−	−	−
$(x^2-2)^2$	+	+	0	+	+	+	+	+	0	+	+
$h'(x)$	+	+	0	+	+	•	−	−	0	−	−
$h(x)$	↗		↗				↘			↘	

$h'(x) = 0 \iff x = 0$ (pt. critico di h)

$x = 0$ pt. di max. loc. stretto ; $\underline{h(0) = 0}$

- $\text{dom}\, h'' = \mathbb{R} \setminus \{-\sqrt{2}, \sqrt{2}\}$ $\quad h''(x) = \dfrac{-4(x^2-2)^{\cancel{2}} + 4x \cdot 2\cancel{(x^2-2)} \cdot 2x}{(x^2-2)^{\cancel{4}\,3}}$

$$= \frac{-4x^2 + 8 + 16x^2}{(x^2-2)^3} = \frac{12x^2+8}{(x^2-2)^3}$$

$$\begin{array}{cccccccccccccc} & + & 0 & - & - & - & - & - & 0 & + & + & & h''(x) \\ -2 & -\sqrt{2} & -1 & & 0 & & 1 & \sqrt{2} & 2 & & & & \\ \cup & & & & \cap & & & & \cup & & & & h(x) \end{array}$$

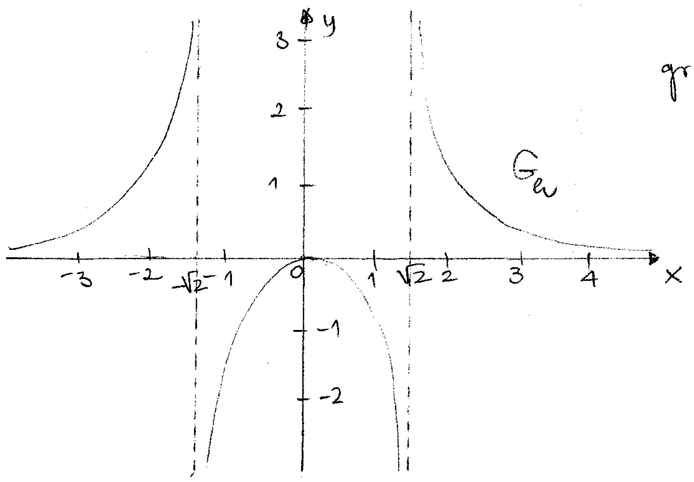


grafico qualitativo di h

2) i) a) $\frac{1}{3} - \frac{1}{3 \cdot 2} + \frac{1}{3 \cdot 3} - \cdots + \frac{1}{3 \cdot 11} = \underline{\underline{\sum_{n=1}^{11} \frac{(-1)^{n+1}}{3 \cdot n}}}$;

$$1 + 5 + 10 + 17 + \cdots + 257 = 1 + (2^2+1) + (3^2+1) + (4^2+1) + \cdots + (16^2+1)$$
$$= \underline{\underline{1 + \sum_{i=2}^{16} (i^2+1)}} \ ;$$

b) $-\frac{1}{3} + \frac{1}{2} - \frac{1}{5} + \frac{1}{4} - \frac{1}{7} + \cdots - \frac{1}{21} + \frac{1}{20} = \underline{\underline{\sum_{n=1}^{10} \left(\frac{-1}{2n+1} + \frac{1}{2n} \right)}}$;

$x + 2x^2 + 3x^3 + \cdots + 12x^{12} = \underline{\underline{\sum_{m=1}^{12} m x^m}}$.

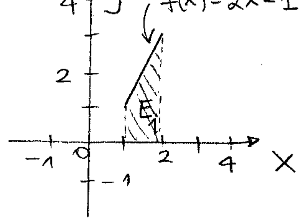
ii) $\sum_{n=1}^{100} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{99} - \cancel{\frac{1}{100}}\right) + \left(\cancel{\frac{1}{100}} - \frac{1}{101}\right)$

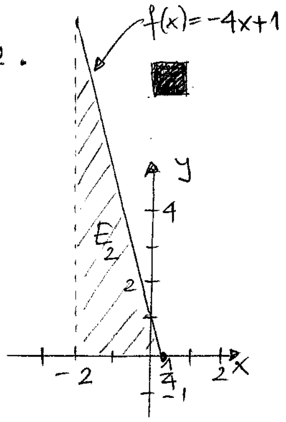
$$= 1 - \frac{1}{101} = \underline{\underline{\frac{100}{101}}} \ ;$$

$\sum_{m=3}^{10} \frac{m}{m-1} = \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9}$.

3) $\int_1^2 (2x-1)\,dx = \text{area } E_1$

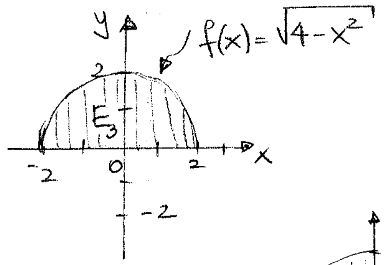
$$= \frac{(1+3)}{2} \cdot 1 = \underline{\underline{2}} \ ;$$



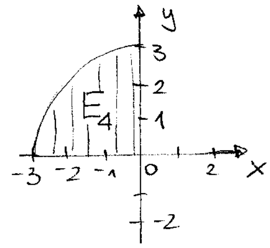


$\int_{-2}^{1/4} (-4x+1)\,dx \underset{\uparrow\ \text{area } E_2}{=} \frac{\left(2+\frac{1}{4}\right) \cdot 9}{2} = \underline{\underline{\frac{81}{8}}}$;

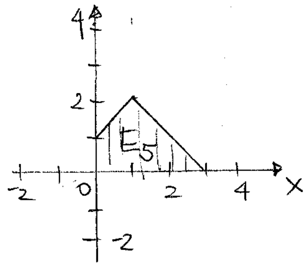
$$\int_{-2}^{2} \sqrt{4-x^2}\,dx = \text{area } E_3 = \frac{\pi \cdot 4}{2} = \underline{\underline{2\pi}} \;;$$



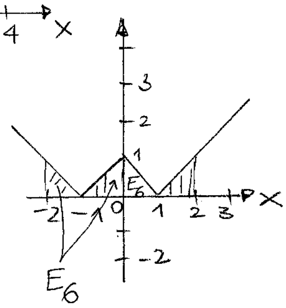
$$\int_{-3}^{0} \sqrt{9-x^2}\,dx \overset{\text{area } E_4}{=} \frac{\pi \cdot 9}{4} = \underline{\underline{\frac{9\pi}{4}}} \;;$$



$$\int_{0}^{3} \left(-|x-1|+2\right) dx = \text{area } E_5 = \underline{\underline{\frac{7}{2}}} \;;$$



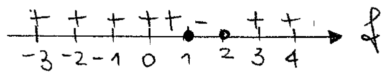
$$\int_{-2}^{2} \big||x|-1\big|\,dx = \text{area } E_6 = \underline{\underline{2}} \;;$$



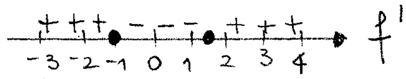
$$\int_{-2}^{2} 3\,dx = \underline{\underline{0}} \,.$$

■

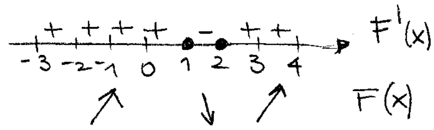
4) i)



ii)



iii) Sappiamo che $F'(x) = f(x)$ e quindi si ha che F è crescente su $[-3, 1]$, e su $[2,4]$, mentre risulta decrescente su $[1,2]$.



Notiamo che $x=1$ è pt. di massimo loc. stretto

$x=2$ è pt. di minimo loc. stretto.

Inoltre abbiamo che $x=0$ è pt. di min. loc. stretto e

$x=4$ è pt. di max. loc. stretto.

■

NOTA: alle conclusioni qui sopra si poteva anche arrivare facendo un grafico approssim. di F (usando la sua definizione)

□

5) i) $\int_{-1}^{-2}(3^x - x)\,dx = \left[\frac{3^x}{\log 3} - \frac{x^2}{2}\right]_{-1}^{-2} = \left(\frac{3^{-2}}{\log 3} - 2\right) - \left(\frac{3^{-1}}{\log 3} - \frac{1}{2}\right) =$

$= \frac{1}{\log 3}\left(\frac{1}{9} - \frac{1}{3}\right) - \frac{3}{2} = -\frac{2}{9\log 3} - \frac{3}{2}$;

$\int_1^e \left(x^{-2} + \frac{1}{x}\right)dx = \left[-x^{-1} + \log|x|\right]_1^e = \left(-\frac{1}{e} + \log e\right) - \left(-1 + \log 1\right) = -\frac{1}{e} + 2$;

$\int_0^2 \frac{x^2-1}{x+1}\,dx = \int_0^2 \frac{(x-1)\cancel{(x+1)}}{\cancel{x+1}}\,dx = \left[\frac{x^2}{2} - x\right]_0^2 = \left(\frac{4}{2} - 2\right) = 0$; $\square$

ii) $\int_2^3 \left(\sqrt[3]{x} - \sqrt[4]{x}\right)dx = \left[\frac{x^{4/3}}{4/3} - \frac{x^{5/4}}{5/4}\right]_2^3 = \left(\frac{3}{4}\cdot 3^{4/3} - \frac{4}{5}\cdot 3^{5/4}\right) - \left(\frac{3}{4}\cdot 2^{4/3} - \frac{4}{5}\cdot 2^{5/4}\right)$;

$\int_{-2}^{-1}\left(\frac{x^2 - 3x^4}{x^4}\right)dx = \int_{-2}^{-1}\left(\frac{1}{x^2} - 3\right)dx = \left[-x^{-1} - 3x\right]_{-2}^{-1} = (1+3) - \left(\frac{1}{2} + 6\right) = -\frac{5}{2}$;

$\int_0^2 (e^x + 1)\,dx = \left[e^x + x\right]_0^2 = (e^2 + 2) - 1 = e^2 + 1$; $\square$

iii) $\int_0^2 (e^x+1)^2\,dx = \int_0^2 (e^{2x} + 2e^x + 1)\,dx = \left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^2 =$

$= \left(\frac{e^4}{2} + 2e^2 + 2\right) - \left(\frac{1}{2} + 2\right) = \frac{e^4}{2} + 2e^2 - \frac{1}{2}$;

$\int_0^1 \left((3x-1)^4 + \frac{1}{2x+1}\right)dx = \left[\frac{1}{3}\cdot\frac{(3x-1)^5}{5} + \frac{1}{2}\log|2x+1|\right]_0^1 =$

$= \left(\frac{1}{15}\cdot 2^5 + \frac{1}{2}\log 3\right) - \left(\frac{-1}{15} + \cancel{\frac{1}{2}\log 1}\right) =$

$= \frac{1}{15}\cdot 2^5 + \frac{1}{2}\log 3 + \frac{1}{15}.$

$\int_0^4 \frac{2x}{x^2+1}\,dx = \left[\log|x^2+1|\right]_0^4 = \log 17$; $\square$

iv) $\int_{-2}^{-1} \frac{x^4+1}{x}\,dx = \int_{-2}^{-1}\left(x^3 + \frac{1}{x}\right)dx = \left[\frac{x^4}{4} + \log|x|\right]_{-2}^{-1} = \left(\frac{1}{4} + \cancel{\log 1}\right) -$

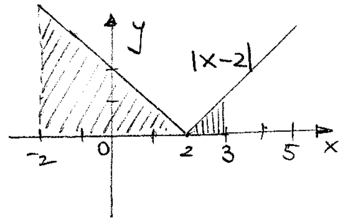
$-\left(\frac{16}{4} + \log|-2|\right) = \frac{1}{4} - 4 - \log 2 = -\frac{15}{4} - \log 2$;

$$\int_0^1 x e^{x^2} dx = \frac{1}{2}\int_0^1 2x e^{x^2} dx = \frac{1}{2}\left[e^{x^2}\right]_0^1 = \underline{\underline{\frac{1}{2}\left(e-1\right)}};$$

$$\int_1^2 \frac{e^{-x}+1}{2}\, dx = \frac{1}{2}\int_1^2 \left(e^{-x}+1\right) dx = \frac{1}{2}\left[-e^{-x}+x\right]_1^2 = \frac{1}{2}\left[\left(-e^{-2}+2\right)-\left(-e^{-1}+1\right)\right]$$

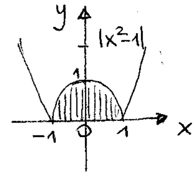
$$= \frac{1}{2}\left(-\frac{1}{e^2}+2+\frac{1}{e}-1\right) = \underline{\underline{\frac{1}{2}\left(-\frac{1}{e^2}+\frac{1}{e}+1\right)}};$$

v) $$\int_{-2}^{3} 3|x-2|\, dx = 3\int_{-2}^{3}|x-2|\, dx =$$



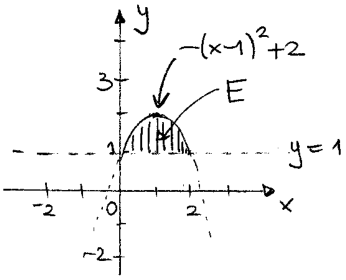
$$= 3\left(\frac{4\cdot 4}{2}+\frac{1}{2}\right) = 3\cdot\frac{17}{2} = \underline{\underline{\frac{51}{2}}};$$

$$\int_{-1}^{1}|x^2-1|\, dx = \int_{-1}^{1}\left(-x^2+1\right) dx = \left[-\frac{x^3}{3}+x\right]_{-1}^{1}$$



$$= \left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right) = -\frac{2}{3}+2 = \underline{\underline{\frac{4}{3}}}$$

6) i)

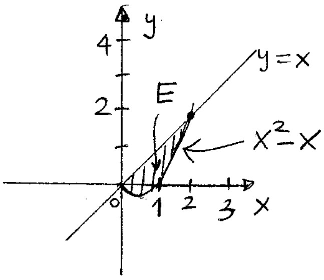


$$\text{area } E = \int_0^2 \left(-(x-1)^2+2-1\right) dx$$

$$= \int_0^2 \left(-x^2+2x\right) dx = \left[-\frac{x^3}{3}+x^2\right]_0^2$$

$$= -\frac{8}{3}+4 = \underline{\underline{\frac{4}{3}}};$$

ii)



$$f(x) = x^2-x = \left(x-\frac{1}{2}\right)^2-\frac{1}{4}$$

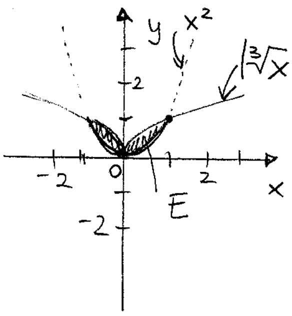
Nota: $x^2-x=x \iff x^2-2x=0$

$x(x-2)=0$

$$\text{area } E = \int_0^2 \left(x-x^2+x\right) dx = \int_0^2\left(-x^2+2x\right) dx = \left[-\frac{x^3}{3}+x^2\right]_0^2$$

$$= \left(-\frac{8}{3}+4\right) = \underline{\underline{\frac{4}{3}}};$$

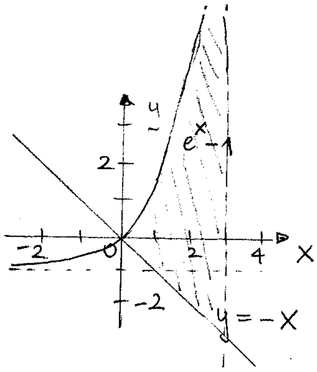
iii)



$$\text{area } E = 2\int_0^1 \left(\sqrt[3]{x}-x^2\right) dx = 2\left[\frac{x^{4/3}}{4/3}-\frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{3}{4}-\frac{1}{3}\right] = \frac{10}{12} = \underline{\underline{\frac{5}{6}}};$$

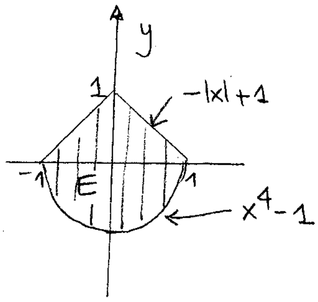
iv)



$$\text{area } E = \int_0^3 \left(e^x - 1 + x\right) dx = \left[e^x - x + \frac{x^2}{2}\right]_0^3 =$$

$$= \left(e^3 - 3 + \frac{9}{2}\right) - (1) = e^3 - 4 + \frac{9}{2} = \underline{\underline{e^3 + \frac{1}{2}}};$$

v)

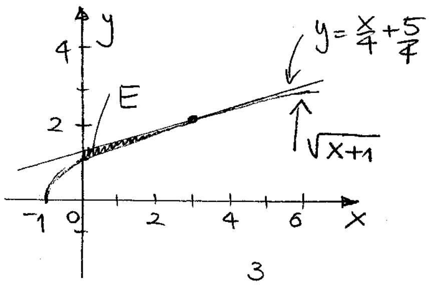


$$\text{area } E = 2\int_0^1 \left(-x + 1 - x^4 + 1\right) dx$$

$$= 2\left[-\frac{x^2}{2} + 2x - \frac{x^5}{5}\right]_0^1 = 2\left[-\frac{1}{2} + 2 - \frac{1}{5}\right]$$

$$= \underline{\underline{\frac{13}{5}}}.$$

7) i)



$$f'(x) = \frac{1}{2\sqrt{x+1}} \qquad f'(3) = \frac{1}{4}$$

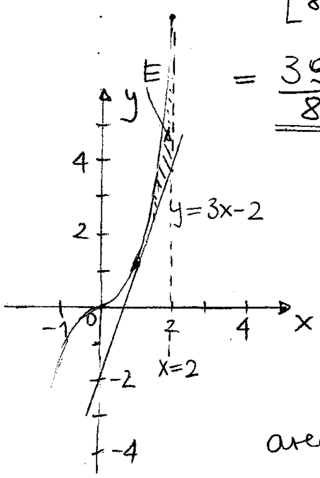
$$y = 2 + \frac{1}{4}(x - 3) \Rightarrow y = \frac{1}{4}x + \frac{5}{4}$$

$$\text{area } E = \int_0^3 \left(\frac{x}{4} + \frac{5}{4} - \sqrt{x+1}\right) dx$$

$$= \left[\frac{x^2}{8} + \frac{5}{4}x - \frac{(x+1)^{3/2}}{3/2}\right]_0^3 = \left(\frac{9}{8} + \frac{15}{4} - \frac{2}{3}\cdot 8\right) + \frac{2}{3}$$

$$= \underline{\underline{\frac{39}{8} - \frac{14}{3}}};$$

ii)



$$f'(x) = 3x^2 \qquad f'(1) = 3$$

$$y = 1 + 3(x - 1) \Rightarrow y = 3x - 2$$

$$\text{area } E = \int_1^2 \left(x^3 - 3x + 2\right) dx = \left[\frac{x^4}{4} - 3\frac{x^2}{2} + 2x\right]_1^2 =$$

$$= \left(\frac{16}{4} - \frac{3}{\cancel{2}}\cdot\overset{2}{\cancel{4}} + 4\right) - \left(\frac{1}{4} - \frac{3}{2} + 2\right)$$

$$= (4 - 6 + 4) - \left(-\frac{5}{4} + 2\right) = \underline{\underline{\frac{5}{4}}}.$$