Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione
CdL in Filosofia
Esame scritto di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2009–2010 – Rovereto, 12 febbraio 2010

FILA Ⓐ

1) i). $A = \{x \in \mathbb{R} : (x+1)|x| < 2\} = \underline{\underline{]-\infty, 1[}}$.

$$(x+1)|x| < 2 \iff \begin{cases} x \geq 0 \\ (x+1)x < 2 \end{cases} \text{ o } \begin{cases} x < 0 \\ (x+1)(-x) < 2 \end{cases}$$

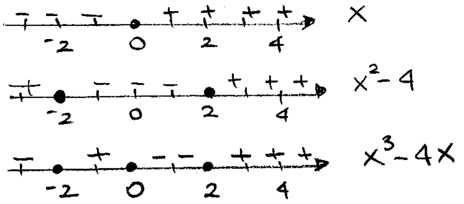
$$\iff \begin{cases} x \geq 0 \\ x^2+x-2 < 0 \end{cases} \text{ o } \begin{cases} x < 0 \\ x^2+x+2 > 0 \end{cases}$$

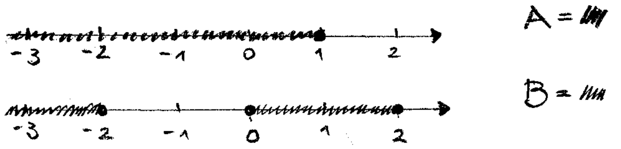
$$\iff \begin{cases} x \geq 0 \\ -2 < x < 1 \end{cases} \text{ o } \begin{cases} x < 0 \\ x \in \mathbb{R} \end{cases}$$

$$\iff \{0 \leq x < 1 \text{ o } \{x < 0$$

$B = \{x \in \mathbb{R} : x^3 - 4x \leq 0\} = \underline{\underline{]-\infty, -2] \cup [0, 2]}}$.

$x^3 - 4x = x(x^2-4)$

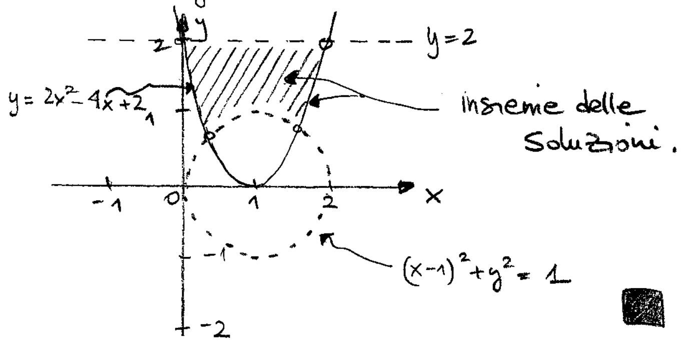




$A \cup B = \underline{\underline{]-\infty, 2]}}$, $A \setminus B = \underline{\underline{]-2, 0[}}$. ☐

ii) $\begin{cases} x^2+y^2-2x > 0 \\ y - 2x^2+4x -2 \geq 0 \\ y < 2 \end{cases} \iff \begin{cases} (x-1)^2+y^2 > 1 \\ y \geq 2(x^2-2x)+2 \\ y < 2 \end{cases}$

$$\iff \begin{cases} (x-1)^2+y^2 > 1 \\ y \geq 2(x-1)^2 \\ y < 2 \end{cases}$$



2) i) $P = (-3,-1)$ $Q = (1,3)$: $\frac{y+1}{3+1} = \frac{x+3}{1+3} \Rightarrow \underline{y = x+2}.$

☐

ii) $y = 3 - 1(x-1)$

$\Rightarrow \underline{\underline{y = -x+4.}}$

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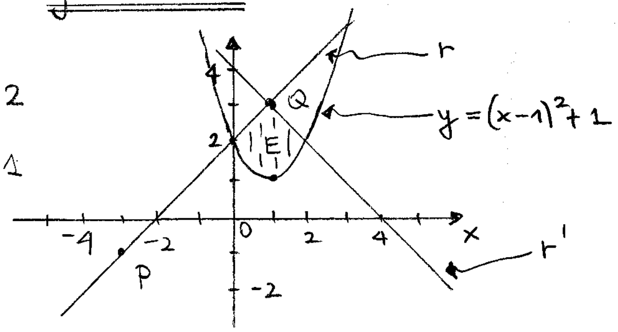
iii) $y = x^2 - 2x + 2$
$= (x-1)^2 + 1$



Si vede facilmente che i pt. d'intersezione della parabola con le rette r e r' sono (0,2) e (2,2). Per simmetria dell'insieme E si ha immediatamente

$$\text{area } E = 2\int_0^1 \left[(x+2) - (x^2-2x+2)\right] dx = 2\int_0^1 (-x^2+3x)\,dx =$$

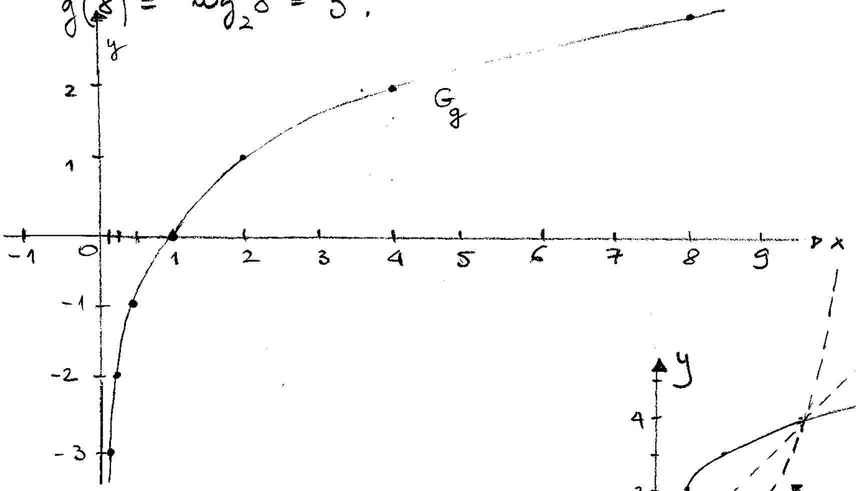
$$= 2\left[-\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^1 = 2\left[-\frac{1}{3}+\frac{3}{2}\right] = \underline{\underline{\frac{7}{3}}}.$$

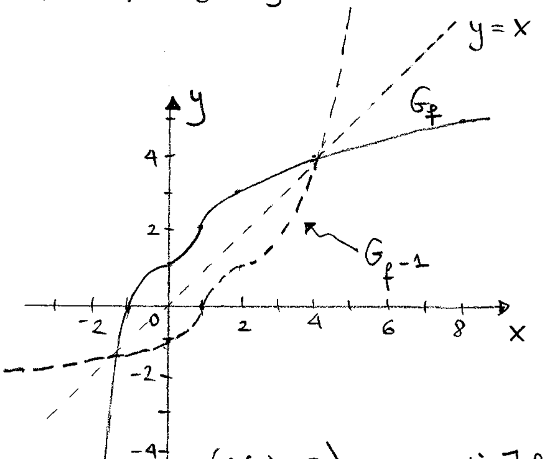
■

3) i) a) $g(\frac{1}{8}) = \log_2 \frac{1}{8} = -3$; $g(\frac{1}{4}) = \log_2 \frac{1}{4} = -2$; $g(\frac{1}{2}) = \log_2 \frac{1}{2} = -1$;
$g(1) = \log_2 1 = 0$; $g(2) = \log_2 2 = 1$; $g(4) = \log_2 4 = 2$;
$g(8) = \log_2 8 = 3$.

☐

b)





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ii) $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^3+1 & \text{se } x \le 0 \\ 2^x & \text{se } 0 < x \le 1 \\ \log_2 x + 2 & \text{se } x > 1 \end{cases}$$

a)

b) f è iniettiva e suriettiva ($f(\mathbb{R}) = \mathbb{R}$); quindi $\exists f^{-1} : \mathbb{R} \to \mathbb{R}$.

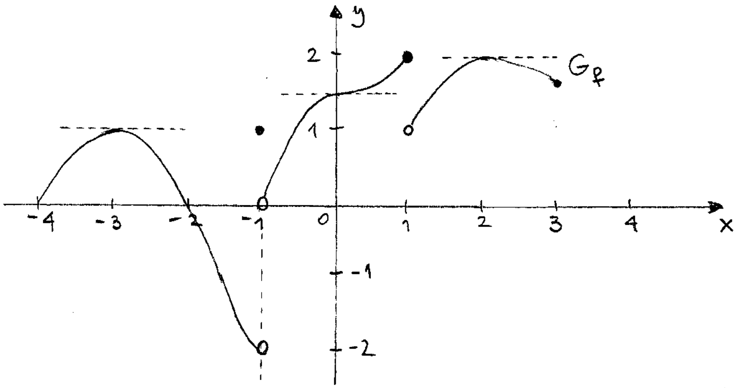
☐

c) $$\sum_{n=1}^{5}\int_{\frac{1}{n+1}}^{\frac{1}{n}} f(x)\,dx = \int_{\frac12}^{1} f(x)\,dx+\int_{\frac13}^{\frac12} f(x)\,dx+\dots+\int_{\frac16}^{\frac15} f(x)\,dx = \int_{\frac16}^{1} f(x)\,dx = \int_{\frac16}^{1} 2^x\,dx = \left[\frac{2^x}{\log 2}\right]_{1/6}^{1}$$

$$= \underline{\underline{\frac{2-\sqrt[6]{2}}{\log 2}}}\ .$$

■

4)



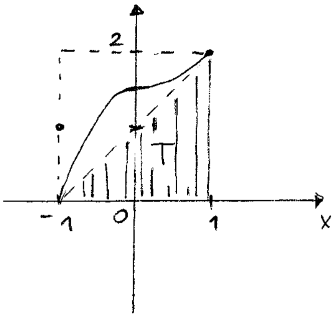
i) $\lim_{x\to -1^-} f(x) = -2$, $\lim_{x\to -1^+} f(x) = 0$, $\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = 1$. ☐

ii) $(-3, 1)$ $y = 1$ eq. della retta tg.

$(0, 3/2)$ $y = 3/2$ "

$(2, 2)$ $y = 2$. " ☐

iii)

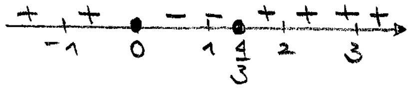


$$\text{area}\,T = \frac{\cancel{2}\cdot 2}{\cancel{2}} \le \int_{-1}^{1} f(x)\,dx \le \text{area}\,Q = 2\cdot 2 = 4$$

dove T è il triangolo $= \{(x,y)\in\mathbb{R}^2 : -1\le x\le 1,\ 0\le y\le x+1\}$

Q è il quadrato $= \{(x,y)\in\mathbb{R}^2 : -1\le x\le 1,\ 0\le y\le 2\}$ ■

5) i) $f(x) = \dfrac{3x^2-4x}{x^2+1}$

- dom $f = \mathbb{R} = \,]-\infty, +\infty[$ (nota: $x^2+1>0\ \forall x\in\mathbb{R}$).
- 

$3x^2-4x$ (segno di f poiché $x^2+1>0$ $\forall x\in\mathbb{R}$).
- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 3$ $y=3$ asintoto orizzontale per f (per $x\to-\infty$, per $x\to+\infty$).
- f è rapporto di due funzioni continue; poiché il denominatore non si annulla mai, f è continua su tutto $\mathbb{R}$.

• dom $f' = \mathbb{R}$

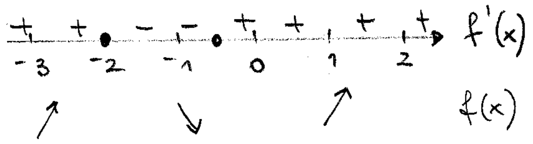
$$f'(x) = \frac{(6x-4)(x^2+1) - (3x^2-4x)\,2x}{(x^2+1)^2}$$

$$= \frac{\cancel{6x^3} + 6x - 4x^2 - 4 - \cancel{6x^3} + 8x^2}{(x^2+1)^2}$$

$$= \frac{4x^2+6x-4}{(x^2+1)^2} = \frac{2(2x^2+3x-2)}{(x^2+1)^2}$$

$2x^2+3x-2=0 \iff$

$$x_{1/2} = \frac{-3 \pm \sqrt{9+16}}{4} = \frac{-3\pm 5}{4} \begin{cases} -2 \\ \frac{1}{2} \end{cases}$$



$x = -2$, $x = \frac{1}{2}$ sono pt. critici di f; dalla monotonia di f segue che $x=-2$ pt. di max. loc. (stretto) per f e $f(-2) = 4$, mentre $x = \frac{1}{2}$ pt. di min. loc (stretto) per f e $f(\frac{1}{2}) = -1$. ☐

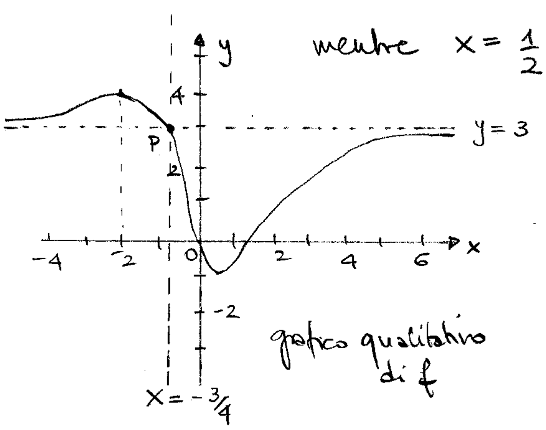


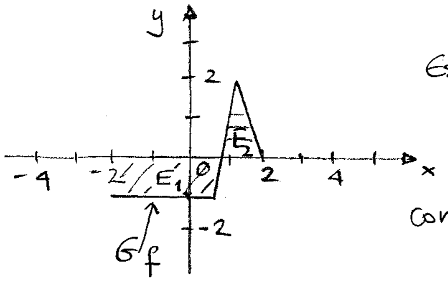
grafico qualitativo di f

ii) f è limitata, poiché $-1 \le f(x) \le 4 \ \forall x \in \mathbb{R}$. ☐

iii) $f(x) = 3 \iff 3x^2 - 4x = 3(x^2+1) \iff x = -\frac{3}{4}$

L'eq. della retta r' è data da $x = -\frac{3}{4}$. ▨

6) i) FALSO!



Esempio di funzione $f : [-2,2] \longrightarrow \mathbb{R}$ continua con $\min_{[-2,2]} f = -1$, $\max_{[-2,2]} f = 2$

$$-\text{area}\, E_1 + \text{area}\, E_2 = \int_{-2}^{2} f(x)\,dx < 0 .$$ ☐

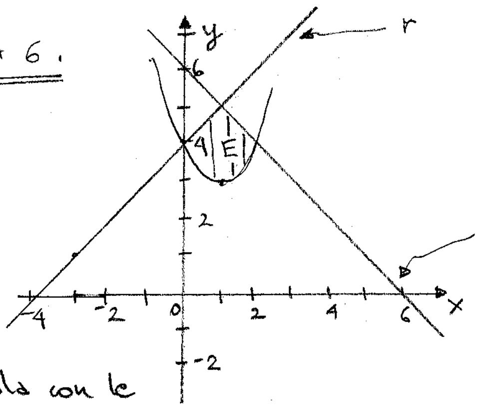
ii) $$C_{24,3} = \frac{24!}{3!\,21!} = \frac{\overset{4}{\cancel{24}} \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{3 \cdot 2} \cdot \cancel{21!}} = \underline{\underline{88 \cdot 23.}}$$ ■

FILA (B)

1) i) $P=(-3,1)$ $\quad Q=(1,5)$: $\quad \frac{y-1}{5-1}=\frac{x+3}{1+3} \Rightarrow \underline{y=x+4.}$ ☐

☐

ii) $y=5-1(x-1) \Rightarrow \underline{\underline{y=-x+6.}}$



iii) $y=x^2-2x+4$

$=(x-1)^2+3$

I pt. di intersezione della parabola con le rette r e r' sono $(0,4)$ e $(2,4)$. Per simmetria dell'insieme E si ha immediatamente

$$\text{area } E = 2\int_0^1\left[(x+4)-(x^2-2x+4)\right]dx = 2\int_0^1(-x^2+3x)\,dx = \underline{\underline{\frac{7}{3}}}.$$

↑ vedi Fila (A) Es. 2 iii)

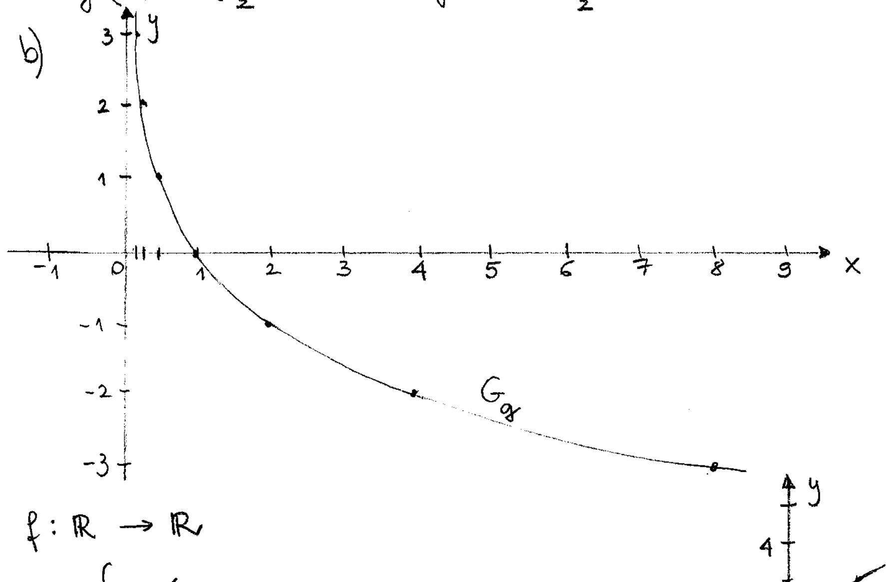
■

2) i) a) $g\left(\frac{1}{8}\right)=\log_{\frac{1}{2}}\frac{1}{8}=3$; $\quad g\left(\frac{1}{4}\right)=\log_{\frac{1}{2}}\frac{1}{4}=2$;

$g\left(\frac{1}{2}\right)=\log_{\frac{1}{2}}\frac{1}{2}=1$; $\quad g(1)=\log_{\frac{1}{2}}1=0$; $\quad g(2)=\log_{\frac{1}{2}}2=-1$;

$g(4)=\log_{\frac{1}{2}}4=-2$; $\quad g(8)=\log_{\frac{1}{2}}8=-3$. ☐

b)

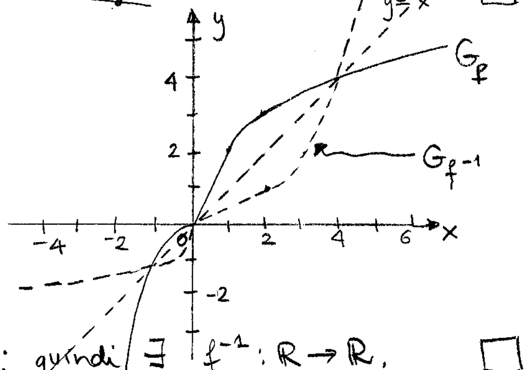


☐

ii) $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}-x^4 & \text{se } x\le 0\\ 2x & \text{se } 0<x\le 1\\ -\log_{\frac{1}{2}}x+2 & \text{se } x>1\end{cases}$$

a)

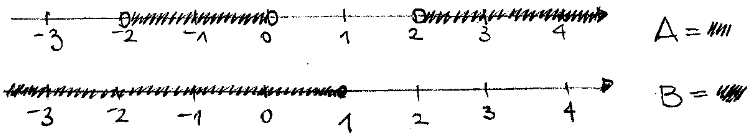


b) f è iniettiva e suriettiva $(f(\mathbb{R})=\mathbb{R})$; quindi $\exists\, f^{-1}:\mathbb{R}\to\mathbb{R}$. ☐

c) $$\sum_{n=1}^{5}\int_{\frac{1}{n+1}}^{\frac{1}{n}}f(x)\,dx=\int_{\frac{1}{6}}^{1}f(x)\,dx=\int_{\frac{1}{6}}^{1}2x\,dx=\left[x^2\right]_{1/6}^{1}=1-\frac{1}{36}=\underline{\underline{\frac{35}{36}}}.$$ ■

3) i) $A=\{x\in\mathbb{R} : x^3-4x>0\} = \underline{\underline{]-2,0[\,\cup\,]2,+\infty[}}$. (vedi Fila Ⓐ Es. 1) i segno di x^3-4x per la determinazione di B).

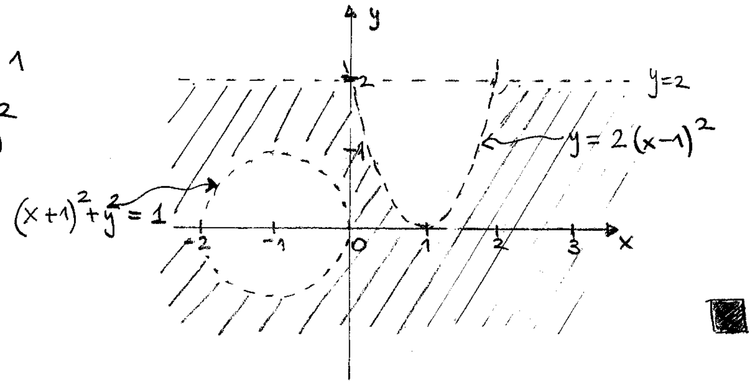
$B=\{x\in\mathbb{R} : (x+1)|x|\le 2\} = \underline{\underline{]-\infty,1]}}$. (vedi Fila Ⓐ, Es. 1) i studio di A).



$A\cup B = \underline{\underline{]-\infty,1]\cup]2,+\infty[}}$, $A\cap B = \underline{\underline{]-2,0[}}$. ☐

ii) $\begin{cases} x^2+y^2+2x>0 \\ y-2x^2+4x-2<0 \\ y<2 \end{cases} \iff \begin{cases} (x+1)^2+y^2>1 \\ y<2(x^2-2x)+2 \\ y<2 \end{cases}$

$\iff \begin{cases} (x+1)^2+y^2>1 \\ y<2(x-1)^2 \\ y<2 \end{cases}$



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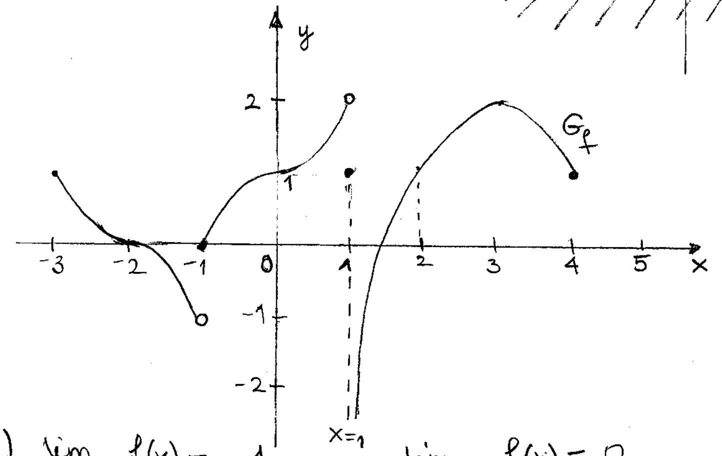
4)



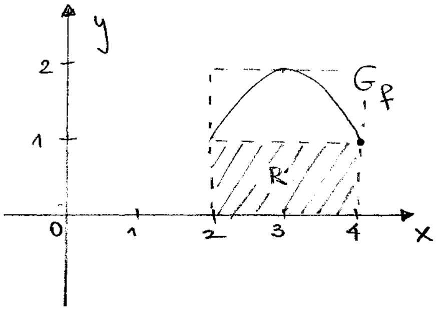
i) $\lim_{x\to -1^-} f(x) = -1$, $\lim_{x\to -1^+} f(x) = 0$, $\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = -\infty$. ☐

ii)

$(-2,0)$	$y=0$	eq. della retta tg.
$(0,1)$	$y=1$	"
$(3,2)$	$y=2$	"

☐

iii)

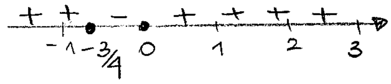


$$\text{area} R = 2\cdot 1 \le \int_2^4 f(x)\,dx \le \text{area}\ Q = 2\cdot 2 = 4$$

dove R è il rettangolo $= \{(x,y)\in\mathbb{R}^2 : 2\le x\le 4,\ 0\le y\le 1\}$

Q è il quadrato $= \{(x,y)\in\mathbb{R}^2 : 2\le x\le 4,\ 0\le y\le 2\}$ ■

5) i) $f(x) = \dfrac{4x^2+3x}{x^2+1}$

- $\text{dom} f = \mathbb{R} =]-\infty, +\infty[$ (nota: $x^2+1 > 0$ $\forall x\in\mathbb{R}$).
- 

$4x^2+3x$ (segno di f poichè $x^2+1>0$ $\forall x\in\mathbb{R}$).
- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 4$

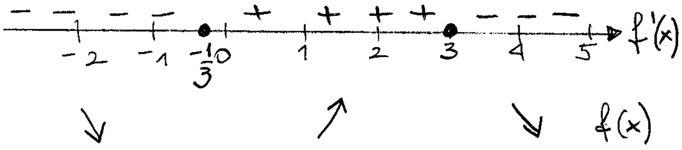
  $y = 4$ asintoto orizzontale per f (per $x\to-\infty$, per $x\to+\infty$).
- f è rapporto di due funzioni continue; poichè il denominatore non si annulla mai, f è continua su tutto $\mathbb{R}$.
- $\text{dom} f' = \mathbb{R}$ $\quad f'(x) = \dfrac{(8x+3)(x^2+1)-(4x^2+3x)2x}{(x^2+1)^2}$

$$= \frac{\cancel{8x^3}+8x+3x^2+3-\cancel{8x^3}-6x^2}{(x^2+1)^2}$$

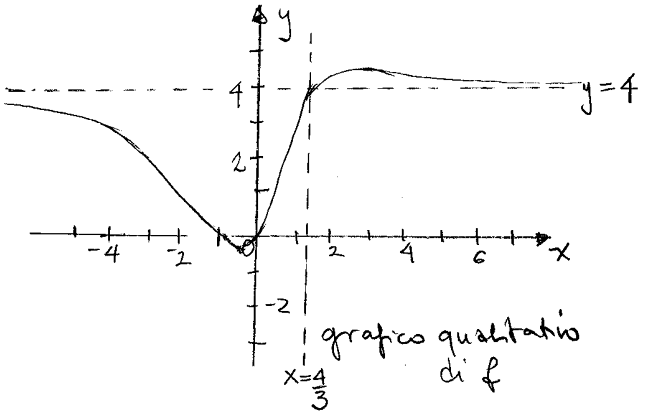
$$= \frac{-3x^2+8x+3}{(x^2+1)^2}$$

$$-3x^2+8x+3=0$$

$$x_{1/2} = \frac{-8\pm\sqrt{64+36}}{-6} = \frac{-8\pm 10}{-6} = \begin{cases} -\frac{1}{3} \\ 3 \end{cases}$$



$x = -\frac{1}{3}$, $x = 3$ sono pt. critici di f; dalla monotonia di f segue che $x = -\frac{1}{3}$ pt. di min. loc. (stretto) per f e $f(-\frac{1}{3}) = -\frac{1}{2}$, mentre $x = 3$ pt. di max. loc. (stretto) per f e $f(3) = \frac{9}{2}$. □



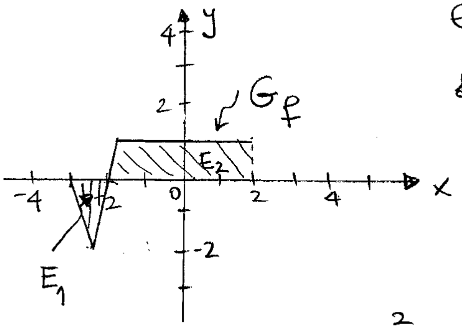
ii) f è limitata, poiché

$$-\frac{1}{2} \le f(x) \le \frac{9}{2} \quad \forall x \in \mathbb{R}.$$ ☐

iii) $f(x)=4 \iff 4x^2+3x = 4(x^2+1)$

$\iff x = \frac{4}{3}.$ ■

6) i) FALSO!



Esempio di funzione

$f : [-3,2] \to \mathbb{R}$ continua

con $\min_{[-3,2]} f = -2$,

$\max_{[-3,2]} f = 1$

$$-\text{area } E_1 + \text{area } E_2 = \int_{-3}^{2} f(x)dx > 0.$$ ☐

ii) $C_{27,4} = \frac{27!}{4!\,23!} = \frac{27\cdot 26\cdot 25\cdot \overset{4}{\cancel{24}}\cdot \cancel{23!}}{\cancel{4\cdot 3\cdot 2}\cdot \cancel{23!}} = \underline{\underline{27\cdot 26\cdot 25}}.$ ■