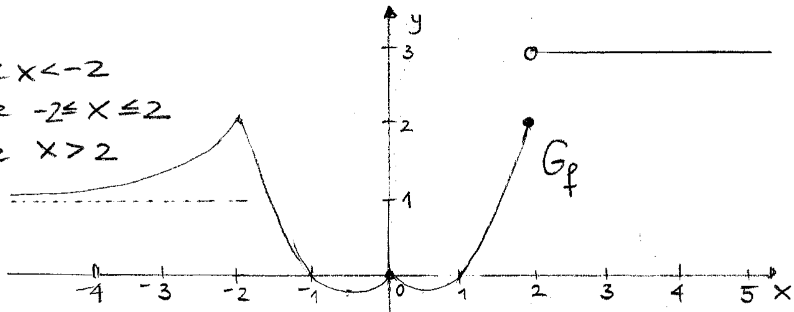
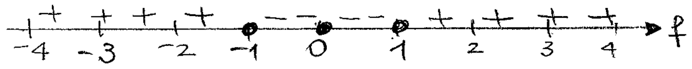
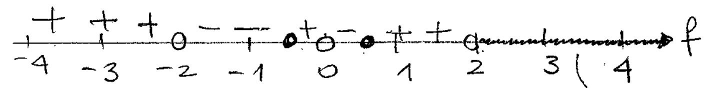
Università degli Studi di Trento — Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia cognitiva
CdL in Interfacce e Tecnologie della Comunicazione
ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2009/10 — Rovereto, 30 novembre — 4 dicembre 2009

1) $f(x)=\begin{cases} e^{x+2}+1 & \text{se } x<-2 \\ x^2-|x| & \text{se } -2\le x\le 2 \\ 3 & \text{se } x>2 \end{cases}$



i)


ii)


→ identicamente $=0$!

iii) $2=$ max loc. per f, $x=-2$ pt. di max loc. stretto; $0=$ max loc. per f, $x=0$ pt. di max loc. stretto per f;

$3=\max\limits_{\mathbb{R}} f$ massimo di f su $\mathbb{R}$, $x\in\,]2,+\infty[$ pt. di massimo.

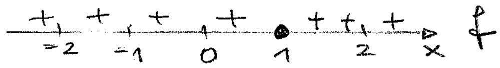
$-\frac{1}{4}=$ min loc. di f, $x\in\{-\frac{1}{2},\frac{1}{2}\}$ pt. di min. loc. per f
(anche $\min\limits_{\mathbb{R}} f=-\frac{1}{4}$)

$x=0$

iv) $x=2$ pt. di discontinuità di f.

2) $\boxed{f(x)=(1-x^3)^2}$

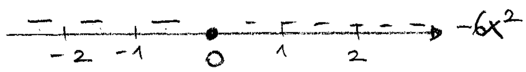
- $\operatorname{dom} f=\mathbb{R}=\,]-\infty,+\infty[$
- segno di f



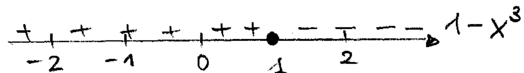
- $\lim\limits_{x\to-\infty} f(x)=+\infty$ $\qquad \lim\limits_{x\to+\infty} f(x)=+\infty$
- f è continua su tutto $\mathbb{R}$ essendo composizione di funzioni continue;
- $\operatorname{dom} f'=\mathbb{R}$ $\quad f'(x)=2(1-x^3)(-3x^2)=-6x^2(1-x^3)$

$f'(x)=0 \iff x=0,\ x=1$ sono pt. critici per f

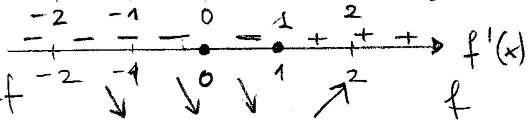
$f(0)=1 \quad f(1)=0$



$x=0$ pt. con tangente orizzontale



$x=1$ pt. di min loc. stretto per f



- dom $f'' = \mathbb{R}$ $\quad f''(x) = -12x(1-x^3) - 6x^2(-3x^2)$

$$= -12x + 12x^4 + 18x^4$$

$$= -12x + 30x^4 = 6x(-2+5x^3)$$

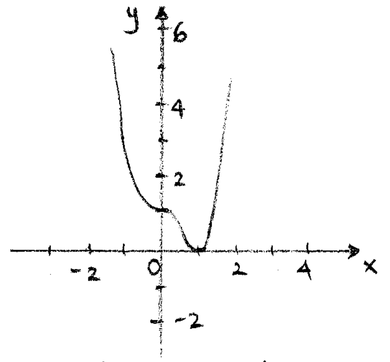
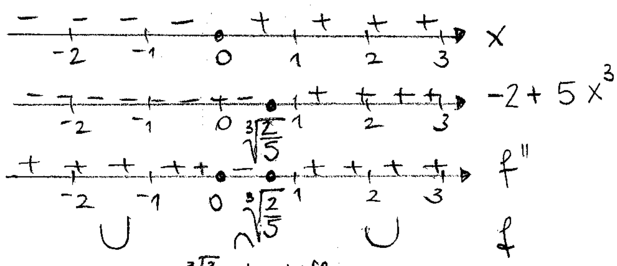


grafico qualitativo di f



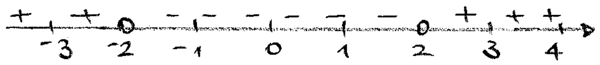
$x=0$ e $x=\sqrt[3]{\frac{2}{5}}$ pt. di flesso.

$\square$

$$\boxed{g(x) = \frac{1}{x^2-4}}$$

g è pari!!

- dom $g = \mathbb{R}\setminus\{-2,2\} = \,]-\infty,-2[\,\cup\,]-2,2[\,\cup\,]2,+\infty[$
- segno di g



- $\lim_{x\to-\infty} g(x) = 0$ $\quad y=0$ asintoto orizzontale per g per $x\to-\infty$;

$\lim_{x\to-2^-} g(x) = +\infty$ $\quad x=-2$ asintoto verticale per g

$\lim_{x\to-2^+} g(x) = -\infty$

$\lim_{x\to 2^-} g(x) = -\infty$ $\quad x=2$ asintoto verticale per g

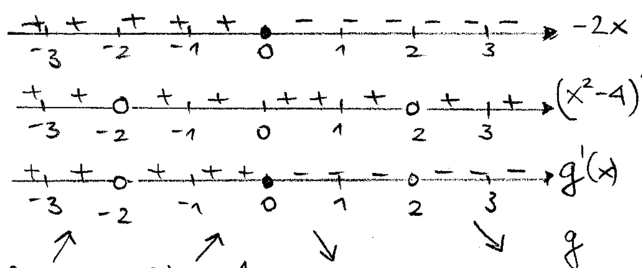
$\lim_{x\to 2^+} g(x) = +\infty$

$\lim_{x\to+\infty} g(x) = 0$ $\quad y=0$ asintoto orizzontale per g per $x\to+\infty$.

- g è continua in tutti i pt. dell'insieme di definizione poiché è rapporto di due funzioni e x^2-4 si annulla solo per $x=-2$ e $x=2$.

- dom $g' = \mathbb{R}\setminus\{-2,2\}$ $\quad g'(x) = \dfrac{-2x}{(x^2-4)^2}$

$g'(x)=0 \iff x=0$ pt. critico per g



$x=0$ pt. di max. loc. e stretto per f $\quad g(0) = -\frac{1}{4}$

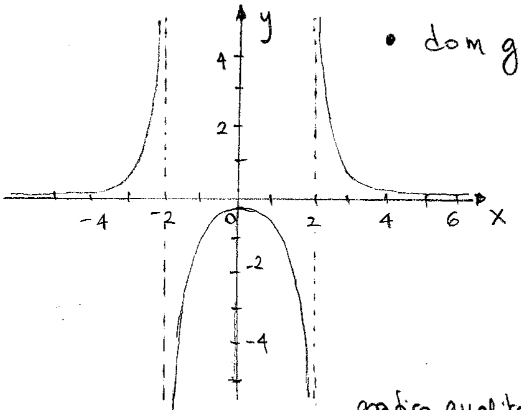


grafico qualitativo di g

• $\mathrm{dom}\, g'' = \mathbb{R}\setminus\{-2,2\}$

$$g''(x) = \frac{-2(x^2-4)^{\cancel{2}} + 2x\cdot 2\cancel{(x^2-4)}2x}{(x^2-4)^{\cancel{4}\,3}}$$

$$= \frac{-2x^2+8+8x^2}{(x^2-4)^3} = \frac{6x^2+8}{(x^2-4)^3} \qquad 6x^2+8>0$$

$$\begin{array}{cccccccccc} + & + & + & 0 & - & - & - & - & 0 & + \; + \\ & -3 & & -2 & -1 & 0 & 1 & & 2 & 3 \\ & \cup & & & & \cap & & & & \cup \end{array} \quad \begin{array}{l} g''(x) \\ \\ g(x) \end{array}$$

Nessun pt. di flesso.

$\square$

$$\boxed{h(x) = \log\left(\frac{x^2}{x^2+1}\right)}$$

h è pari!!

• $\mathrm{dom}\, h = \mathbb{R}\setminus\{0\} =]-\infty,0[\,\cup\,]0,+\infty[$

• segno di h : osserviamo che $0 < \frac{x^2}{x^2+1} < 1$

per ogni $x\in\mathbb{R}\setminus\{0\}$ e quindi $h(x) = \log\left(\frac{x^2}{x^2+1}\right) < 0$ $\forall x \in \mathbb{R}\setminus\{0\}$.

• $\lim_{x\to-\infty} h(x) = \log 1 = 0$ $\qquad y=0$ asintoto orizzont. per h, per $x\to-\infty$

$\lim_{x\to 0^-} h(x) = -\infty$ $\qquad x=0$ asintoto verticale per h

$\lim_{x\to 0^+} h(x) = -\infty$

$\lim_{x\to+\infty} h(x) = 0$ $\qquad y=0$ asintoto orizzontale per h, per $x\to+\infty$.

• h è continua in tutti i pt. di $\mathbb{R}\setminus\{0\}$ essendo composizione di funzioni continue su $\mathbb{R}\setminus\{0\}$.

• $\mathrm{dom}\, h' = \mathbb{R}\setminus\{0\}$ $\qquad h'(x) = \frac{1}{\left(\frac{x^2}{x^2+1}\right)}\cdot\left(\frac{x^2}{x^2+1}\right)'$

$$h'(x) = \frac{\cancel{x^2+1}}{x^2}\cdot\frac{2x(x^2+1)-x^2\cdot 2x}{(x^2+1)^{\cancel{2}}} = \frac{2x}{x^2(x^2+1)}$$

$$= \frac{2}{x(x^2+1)} \qquad (x^2+1)>0$$

$$\begin{array}{ccccccccc} - & - & - & - & 0 & + & + & + & + \\ -2 & & -1 & & 0 & & 1 & & 2 \\ & & \searrow & & & & & \nearrow & \end{array} \quad \begin{array}{l} h'(x) \\ \\ h(x) \end{array}$$

$\nexists$ pt. critico per h

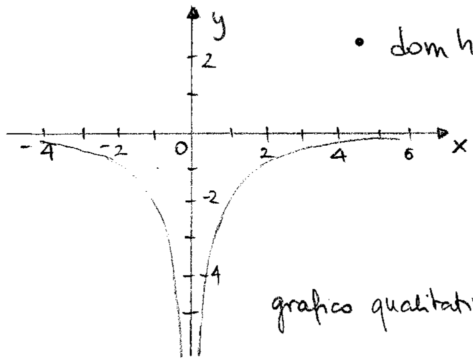


grafico qualitativo di h

• $\operatorname{dom} h'' = \mathbb{R}\setminus\{0\}$ $\quad h''(x) = \dfrac{-2\left[(x^2+1)+x\cdot 2x\right]}{x^2(x^2+1)^2}$

$$= \frac{-2\,(3x^2+1)}{x^2(x^2+1)^2}$$

$- \; - \; - \; - \; \overset{\circ}{0} \; - \; - \; - \; - \; \to \; h''(x)$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

$\frown \qquad\qquad \frown \qquad h(x)$

$\square$

$$\boxed{v(x) = (x^2+2x)e^{-x}} \qquad \left(= \frac{x^2+2x}{e^x}\right)$$

• $\operatorname{dom} v = \mathbb{R} = \,]-\infty, +\infty[$ $\qquad e^x > 0 \;\; \forall x \in \mathbb{R}$

• segno di v $\qquad + \; + \; \bullet \; - \; - \; - \; \bullet \; + \; + \; + \; \to \; v$

$-3 \quad -2 \quad -1 \quad 0 \quad 1$

• $\lim_{x\to-\infty} v(x) = +\infty$ $\qquad \lim_{x\to+\infty} v(x) = 0$ $\qquad y=0$ asintoto orizzontale per v per $x\to+\infty$

• v è rapporto di funzioni continue e il denominatore non si annulla mai; quindi v è continua su $\mathbb{R}$.

• $\operatorname{dom} v' = \mathbb{R}$ $\qquad v'(x) = (2x+2)e^{-x} + (x^2+2x)e^{-x}(-1)$

$$= e^{-x}\left[\cancel{2x}+2-x^2-\cancel{2x}\right]$$

$$= e^{-x}\left[2-x^2\right]$$

$x_{1,2} = \pm\sqrt{2}$ pt. critici per v

$- \; - \; \bullet \; + \; + \; + \; + \; \bullet \; - \; - \; - \; - \; \to \; v'(x)$

$-2 \; -\sqrt{2} \; -1 \quad 0 \quad 1 \; \sqrt{2} \; 2 \quad 3$

$\searrow \qquad \nearrow \qquad \searrow \qquad v(x)$

$x = -\sqrt{2}$ pt. di min. loc. stretto per v

$$v(-\sqrt{2}) = (2-2\sqrt{2})\,e^{+\sqrt{2}} = 2(1-\sqrt{2})\,e^{\sqrt{2}} \qquad (\sim -3.4)$$

$x = \sqrt{2}$ pt. di max. loc. stretto per v

$$v(\sqrt{2}) = (2+2\sqrt{2})\,e^{-\sqrt{2}} = \frac{2(1+\sqrt{2})}{e^{\sqrt{2}}} \qquad (\sim 1.2)$$

• $\operatorname{dom} v'' = \mathbb{R}$ $\qquad v''(x) = -2x e^{-x} + (2-x^2)e^{-x}(-1)$

$$= e^{-x}(x^2-2x-2)$$

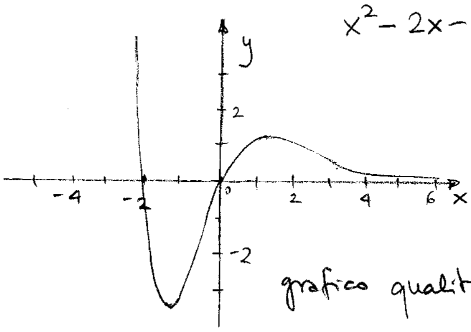
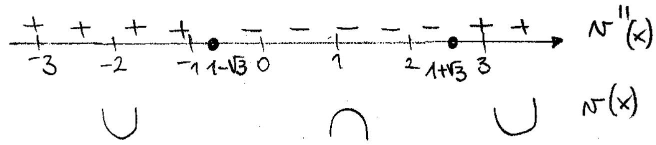


grafico qualitativo di v

$$x^2-2x-2=0 \iff x_{1/2}=\frac{2\pm\sqrt{4+8}}{2}=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$



pt. di flesso $x=1-\sqrt{3}$
$x=1+\sqrt{3}$.

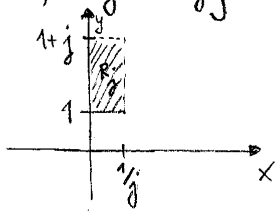
3) i) a) $$-\frac{1}{2}+\frac{1}{4}-\frac{1}{6}+\dots-\frac{1}{54}=-\frac{1}{2\cdot1}+\frac{1}{2\cdot2}-\frac{1}{2\cdot3}+\dots-\frac{1}{2\cdot27}=\underline{\underline{\sum_{n=1}^{27}(-1)^n\frac{1}{2n}}}.$$

$$\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\dots+\frac{16}{257}=\frac{1}{1+1}+\frac{2}{2^2+1}+\frac{3}{3^2+1}+\dots+\frac{16}{16^2+1}=\underline{\underline{\sum_{m=1}^{16}\frac{m}{m^2+1}}}.$$

b) $$-\frac{2}{3}+\frac{3}{2}-\frac{2}{5}+\frac{3}{4}-\frac{2}{7}+\dots-\frac{2}{21}+\frac{3}{20}=-\frac{2}{2+1}+\frac{3}{2}-\frac{2}{2\cdot2+1}+\frac{3}{2\cdot2}-\frac{2}{2\cdot3+1}+\dots$$
$$-\frac{2}{2\cdot10+1}+\frac{3}{2\cdot10}=\underline{\underline{\sum_{n=1}^{10}\left(-\frac{2}{2n+1}+\frac{3}{2n}\right)}}.$$

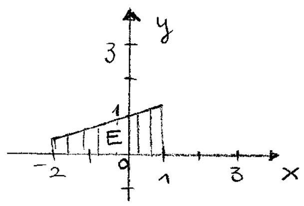
$$x^2+x^4+x^9+\dots+x^{121}=\underline{\underline{x^2+\sum_{k=2}^{11}x^{k^2}}}$$

ii) a) $$\sum_{j=1}^{6}\text{area}(R_j)\qquad R_j=\left\{(x,y)\in\mathbb{R}^2:\ 0\le x\le\frac{1}{j},\ 1\le y\le 1+j\right\}$$
$$=\sum_{j=1}^{6}\frac{1}{j}\cdot j=1+1+1+1+1+1=\underline{\underline{6}}$$

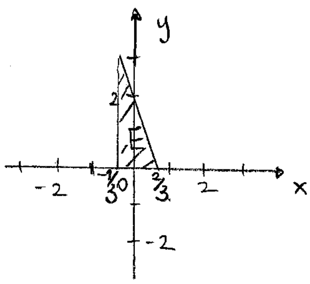


b) $$\sum_{m=3}^{6}\frac{m}{m+1}=\underline{\underline{\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}}}.$$

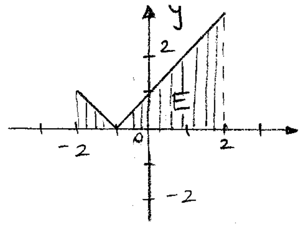
4) $$\int_{-2}^{1}\left(\frac{x}{3}+1\right)dx=\text{area}(E)$$
$$=\frac{\frac{1}{3}+\frac{4}{3}}{2}\cdot3=\underline{\underline{\frac{5}{2}}}$$



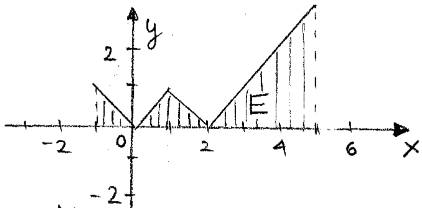
$$\int_{-1/3}^{2/3}(-3x+2)\,dx=\text{area}(E)$$
$$=\frac{1\cdot3}{2}=\underline{\underline{\frac{3}{2}}}$$



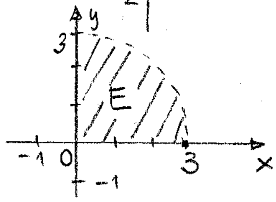
$$\int_{-2}^{2} |x+1|\,dx = \text{area}(E)$$
$$= \frac{1}{2} + \frac{3\cdot 3}{2} = \frac{10}{2} = \underline{\underline{5}}$$



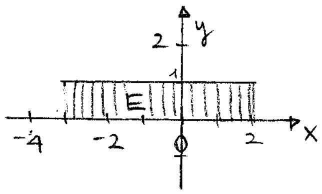
$$\int_{-1}^{5} \big||x-1|-1\big|\,dx = \text{area}(E)$$
$$= \frac{1}{2} + \frac{2\cdot 1}{2} + \frac{3\cdot 3}{2}$$
$$= \frac{12}{2} = \underline{\underline{6}}$$



$$\int_{0}^{3} \sqrt{9-x^2}\,dx = \text{area}(E) =$$
$$= \underline{\underline{\frac{9\pi}{4}}}$$



$$\int_{-3}^{2} dx = \underline{\underline{5}}$$



5) i) $\displaystyle\int_{-2}^{-1} (e^x - x)\,dx = \left[e^x - \frac{x^2}{2}\right]_{-2}^{-1} = \left(e^{-1} - \frac{1}{2}\right) - \left(e^{-2} - 2\right) = \underline{\underline{\frac{1}{e} - \frac{1}{e^2} + \frac{3}{2}}}$;

$\displaystyle\int_{0}^{1} (x^2 - 4^x)\,dx = \left[\frac{x^3}{3} - \frac{4^x}{\log 4}\right]_0^1 = \left(\frac{1}{3} - \frac{4}{\log 4}\right) - \left(-\frac{1}{\log 4}\right) = \underline{\underline{\frac{1}{3} - \frac{3}{\log 4}}}$;

$\displaystyle\int_{2}^{4} \left(x^{-3} - \frac{1}{x}\right)dx = \left[\frac{x^{-2}}{-2} - \log|x|\right]_2^4 = \left(-\frac{1}{32} - \log 4\right) - \left(-\frac{1}{8} - \log 2\right) =$

$\displaystyle= \underline{\underline{\frac{3}{32} - \log 2}}$;

ii) $\displaystyle\int_{3}^{4} \left(\sqrt[3]{x} - \sqrt{x}\right)dx = \int_{3}^{4} \left(x^{1/3} - x^{1/2}\right)dx = \left[\frac{x^{4/3}}{4/3} - \frac{x^{3/2}}{3/2}\right]_3^4 = \left(\frac{3\sqrt[3]{4^4}}{4} - \frac{2(\sqrt{4})^3}{3}\right)$

$\displaystyle- \left(\frac{3}{4}\sqrt[3]{3^4} - \frac{2(\sqrt{3})^3}{3}\right) = \underline{\underline{\left(3\sqrt[3]{4} - \frac{16}{3}\right) - \left(\frac{9}{4}\sqrt[3]{3} - 2\sqrt{3}\right)}}$

$\displaystyle\int_{1}^{e} \frac{x^3 - 4x^2}{x^3}\,dx = \int_{1}^{e} \left(1 - \frac{4}{x}\right)dx = \left[x - 4\log|x|\right]_1^e = \left(e - 4\log e\right) - \left(1 - 4\cancel{\log 1}\right)$

$\displaystyle= e - 4 - 1 = \underline{\underline{e - 5}}$;

$\displaystyle\int_{0}^{2} (e^x + 1)\,dx = \left[e^x + x\right]_0^2 = \left(e^2 + 2\right) - \left(1\right) = \underline{\underline{e^2 + 1}}$;

iii) $\displaystyle\int_{0}^{1} (e^x + 1)^5 e^x\,dx = \left[\frac{(e^x+1)^6}{6}\right]_0^1 = \frac{(e+1)^6}{6} - \frac{2^6}{6} = \underline{\underline{\frac{(e+1)^6}{6} - \frac{32}{3}}}$

$$\int_0^1 \left((2x+1)^4 + \frac{1}{x+1}\right) dx = \int_0^1 2\left(\frac{2x+1}{2}\right)^4 dx + \int_0^1 \frac{1}{x+1}\, dx$$

$$= \frac{1}{2}\int_0^1 2(2x+1)^4 dx + \Big[\log|x+1|\Big]_0^1 =$$

$$= \frac{1}{2}\left[\left(\frac{2x+1}{5}\right)^5\right]_0^1 + \log 2 = \underline{\underline{\frac{1}{2}\left(\frac{3^5}{5} - \frac{1}{5}\right) + \log 2}}\,;$$

$$\int_1^2 \frac{2x}{x^2+1}\, dx = \Big[\log(x^2+1)\Big]_1^2 = \log 5 - \log 2 = \underline{\underline{\log\frac{5}{2}}}.$$