Università di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
Esame scritto di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2010–2011 – Rovereto, 5 luglio 2011

FILA (A)

1) i) $\mathcal{A}$ = "Non pioverà nei prossimi giorni"

$\mathcal{B}$ = "Il verde pubblico subirà gravi danni".

Allora la proposizione data in matematichese si scrive "$\mathcal{A} \Rightarrow \mathcal{B}$".

Poiché $\mathcal{A} \Rightarrow \mathcal{B} \Leftrightarrow (\text{non}\,\mathcal{A})\text{ o }\mathcal{B}$, allora

$\text{non}\left[\mathcal{A} \Rightarrow \mathcal{B}\right] \Leftrightarrow \text{non}\left[(\text{non}\,\mathcal{A})\text{ o }\mathcal{B}\right] \Leftrightarrow \left[\text{non}(\text{non}\,\mathcal{A})\right]\text{ e }\left[\text{non}\,\mathcal{B}\right]$

$\Leftrightarrow \mathcal{A}\text{ e }(\text{non}\,\mathcal{B})$. La negazione della proposizione data si scrive quindi come segue: "Non pioverà nei prossimi giorni e il verde pubblico non subirà gravi danni". □

ii) a)

$\mathcal{A}$:	V	F
non $\mathcal{A}$:	F	V
$\mathcal{A}$ o (non $\mathcal{A}$) :	V	V

← $\mathcal{A}$ o (non $\mathcal{A}$) è una tautologia.

b)

$\mathcal{A}$:	V	F
non $\mathcal{A}$:	F	V
$\mathcal{A}$ e (non $\mathcal{A}$) :	F	F
non [$\mathcal{A}$ e (non $\mathcal{A}$)] :	V	V

← non [$\mathcal{A}$ e (non $\mathcal{A}$)] è una tautologia. ■

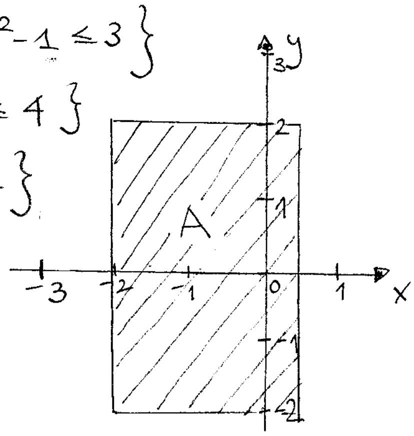
2) i) $A = \{(x,y)\in\mathbb{R}^2 : 2x^2+3x \le 2,\ |y^2-1|\le 3\}$

$= \{(x,y)\in\mathbb{R}^2 : 2x^2+3x-2 \le 0,\ -3 \le y^2-1 \le 3\}$

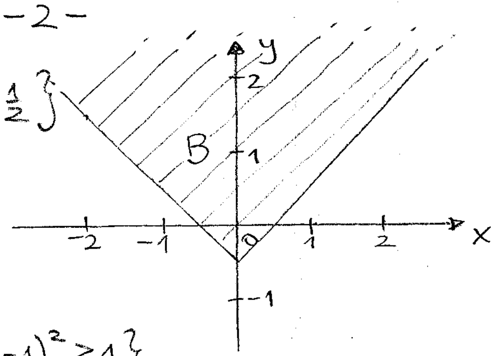
$= \{(x,y)\in\mathbb{R}^2 : -2 \le x \le \frac{1}{2},\ -2 \le y^2 \le 4\}$

$= \{(x,y)\in\mathbb{R}^2 : -2 \le x \le \frac{1}{2},\ -2 \le y \le 2\}$

$= \left[-2,\frac{1}{2}\right]\times\left[-2,2\right]$.

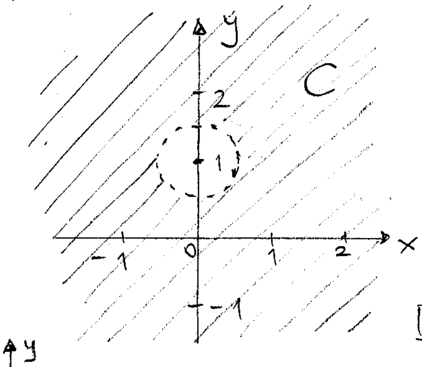


$B = \{(x,y) \in \mathbb{R}^2 : \ y \geq |x| - \frac{1}{2}\}$

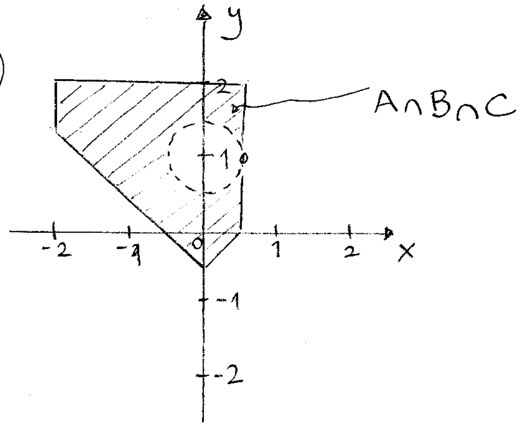


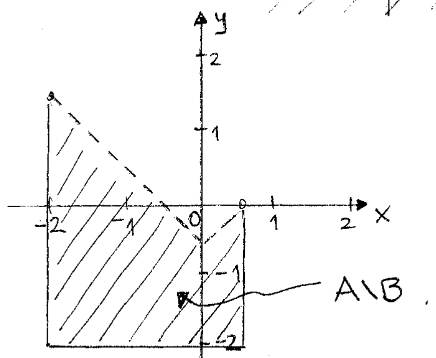
$C = \{(x,y) \in \mathbb{R}^2 : \ 4x^2 + 4(y-1)^2 > 1\}$

$\quad = \{(x,y) \in \mathbb{R}^2 : \ x^2 + (y-1)^2 > \frac{1}{4}\}$



ii)





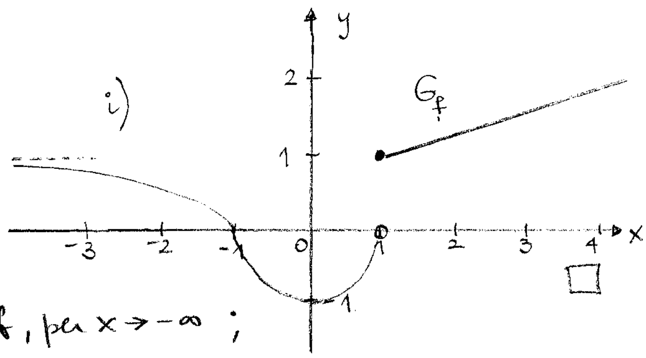
iii) $x = k$ con $k < -2$ o $k > \frac{1}{2}$ non intersecano mai l'insieme A.

$y = k$ con $k < -\frac{1}{2}$ non intersecano mai l'insieme B.

3) $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} -e^{x+1} + 1 & \text{se } x < -1 \\ x^2 - 1 & \text{se } x \in [-1, 1[\\ \frac{x}{3} + \frac{2}{3} & \text{se } x \geq 1 \end{cases}$$

i)



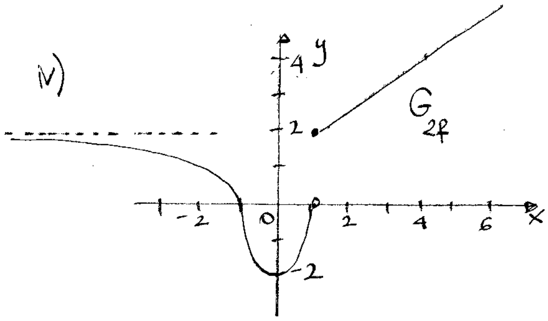
ii) $y = 1$ asintoto orizzontale per f, per $x \to -\infty$;

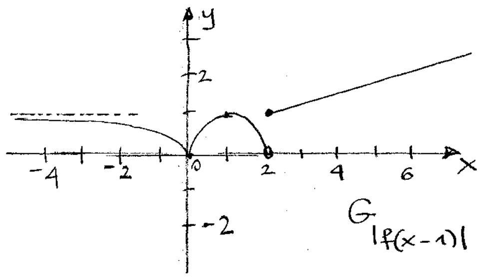
$y = \frac{x}{3} + \frac{2}{3}$ asintoto obliquo per f, per $x \to +\infty$.

iii) $$\sum_{k=0}^{4} f(-2+k) = f(-2) + f(-1) + f(0) + f(1) + f(2) = \left(-\frac{1}{e} + 1\right) + 0 \cancel{-1} + \cancel{1} + \frac{4}{3}$$

$$= \underline{\underline{-\frac{1}{e} + \frac{7}{3}}}.$$

iv)





v) $(f \circ f)(0) = f(f(0)) = f(-1) = 0$.

è ben definito, poiché è definito $f(0)$ e $f(-1)$.

4) $f : \mathbb{R} \to \mathbb{R} \qquad f(x) = (x^2-4)^2$

f è pari!

i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$, e $= 0 \iff x = \pm 2$.

$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$

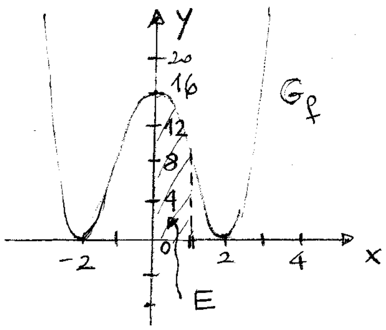
$f'(x) = 4x(x^2-4)$

$$\begin{array}{ccccccccc} & - & & + & & - & & + & f'(x) \\ & \searrow & -2 & \nearrow & 0 & \searrow & 2 & \nearrow \ 4 & f(x) \end{array}$$

$f''(x) = 4(x^2-4) + 8x^2 = 12x^2 - 16 = 4(3x^2-4)$

$f''(x) = 0 \iff x = \pm \frac{2}{\sqrt{3}}$.

$$\begin{array}{cccccc} + & & - \ - & & + & f''(x) \\ \cup \ -2 & & \overset{0}{\cap} & & 2 \ \cup \ 4 & f(x) \end{array}$$



Attenzione! La scala è diversa sull'asse x e sull'asse y.

ii) $f(\mathbb{R}) = [0, +\infty[$.

iii) $\min_{\mathbb{R}} f = 0 \qquad x = -2, \ x = 2$ pt. di minimo.

$\nexists \max_{\mathbb{R}} f$.

iv) $[2, +\infty[$: su questo intervallo non limitato superiormente, f risulta iniettiva.

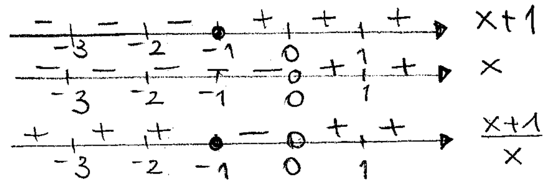
v) $\displaystyle\int_3^5 \frac{f'(x)}{f(x)}\,dx = \Big[\log|f(x)|\Big]_3^5 = \Big[\log (x^2-4)^2\Big]_3^5 = \log 21^2 - \log 5^2 = \underline{\underline{2 \log \frac{21}{5}}}$.

$\displaystyle\int_0^1 f(x)\,dx = \int_0^1 (x^2-4)^2\,dx = \int_0^1 (x^4 - 8x^2 + 16)\,dx = \left[\frac{x^5}{5} - 8\frac{x^3}{3} + 16x\right]_0^1 =$

$= \frac{1}{5} - \frac{8}{3} + 16 = \frac{3 - 40 + 240}{15} = \underline{\underline{\frac{203}{15}}}$.

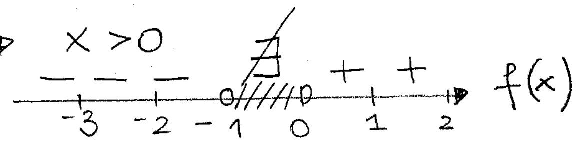
$\displaystyle\int_0^1 f(x)\,dx = $ area E.

5) $f(x) = \log_2\left(\frac{x+1}{x}\right)$ i) • $\mathrm{dom}f = \left\{x \in \mathbb{R} : \frac{x+1}{x} > 0\right\} = \left]-\infty, -1\right[\cup \left]0, +\infty\right[$.



• segno di f : $f(x) > 0 \iff \frac{x+1}{x} > 1 \iff \frac{1}{x} > 0$

$\Rightarrow x > 0$



• $\lim_{x \to -\infty} \log_2\left(\frac{x+1}{x}\right) = \underline{\underline{0}}$ $\quad \frac{x+1}{x} \to 1$

y=0 asint. orizz. per f (per $x \to -\infty$)

$\lim_{x \to -1^-} \log_2\left(\frac{x+1}{x}\right) = \underline{\underline{-\infty}}$ $\quad \frac{x+1}{x} \to$ infinitesimo positivo

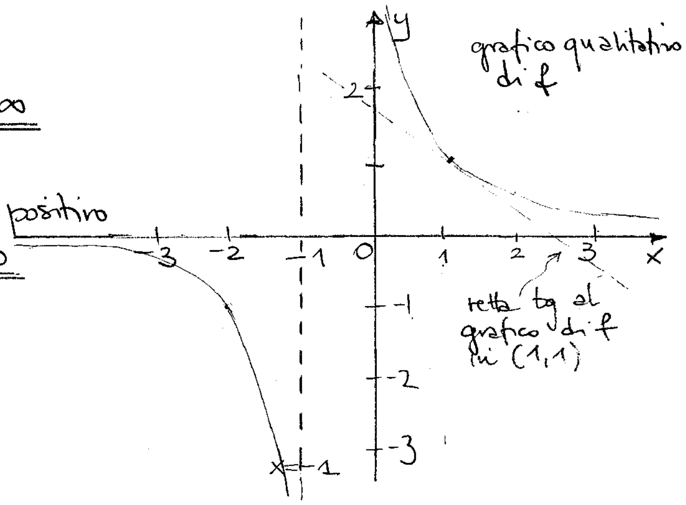
x=−1 asint. verticale per f

$\lim_{x \to 0^+} \log_2\left(\frac{x+1}{x}\right) = \underline{\underline{+\infty}}$ $\quad \frac{x+1}{x} \to +\infty$

x=0 asint. verticale per f

$\lim_{x \to +\infty} \log_2\left(\frac{x+1}{x}\right) = \underline{\underline{0}}$. $\quad \frac{x+1}{x} \to 1$

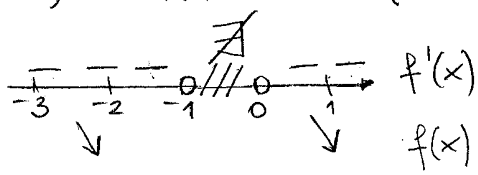
y= 0 asint. orizz. per f (per $x \to +\infty$)



• f è continua, essendo rapporto di funzioni continue e composizione di funzioni continue.

• $f'(x) = \frac{1}{\left(\frac{x+1}{x}\right)\log 2} \cdot \left(\frac{x+1}{x}\right)' = \frac{1}{(\log 2)}\frac{\cancel{x}}{(x+1)} \cdot \frac{x-(x+1)}{x^{\cancel{2}}} = \frac{-1}{x(x+1)} \cdot \frac{1}{(\log 2)}$

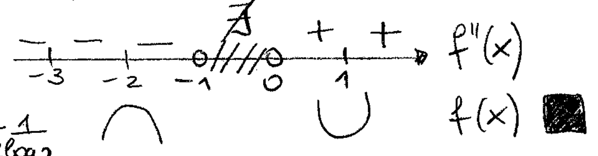
$\mathrm{dom}f' = \mathrm{dom}f$



$\nexists$ sono pt. critici

• $\mathrm{dom}f'' = \mathrm{dom}f$

$f''(x) = \frac{(x+1)+x}{(x^2+x)^2} \cdot \frac{1}{\log 2} = \frac{2x+1}{(x^2+x)^2} \cdot \frac{1}{\log 2}$



ii) $y = 1 - \frac{1}{2\log 2}(x-1) \Rightarrow \underline{\underline{y = -\frac{1}{2\log 2}x + 1 + \frac{1}{2\log 2}}}$

6) $P_4 \cdot P_3 = \underline{\underline{4! \cdot 3!}}$

FILA (B)

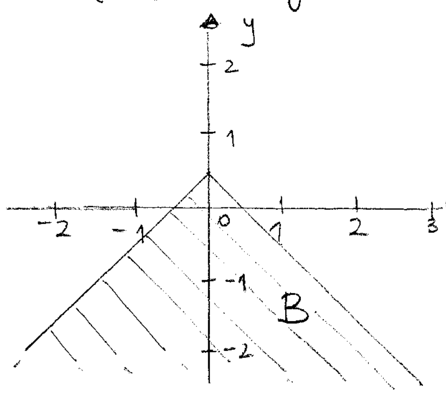
1) i) $A = \{(x,y) \in \mathbb{R}^2 : 2x^2 - 3x \leq 2 \;,\; |y^2 - 1| \leq 3\}$

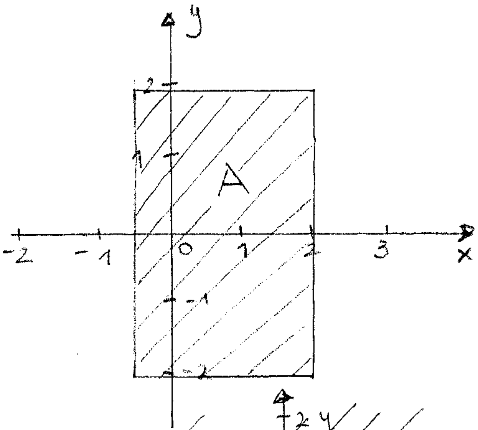
$= \{(x,y) \in \mathbb{R}^2 : 2x^2 - 3x - 2 \leq 0 \;,\; -3 \leq y^2 - 1 \leq 3\}$

$= \{(x,y) \in \mathbb{R}^2 : -\frac{1}{2} \leq x \leq 2 \;,\; -2 \leq y^2 \leq 4\}$

$= \{(x,y) \in \mathbb{R}^2 : -\frac{1}{2} \leq x \leq 2 \;,\; -2 \leq y \leq 2\} = [-\frac{1}{2}, 2] \times [-2, 2]$

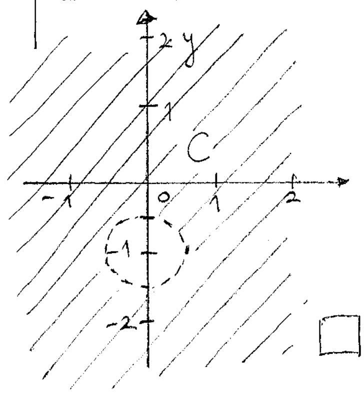
$B = \{(x,y) \in \mathbb{R}^2 : y \leq -|x| + \frac{1}{2}$



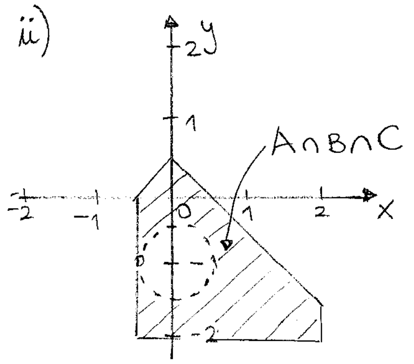


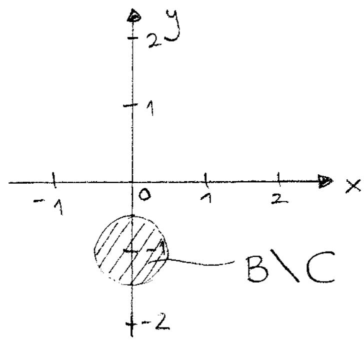
$C = \{(x,y) \in \mathbb{R}^2 : 4x^2 + 4(y+1)^2 > 1\}$

$= \{(x,y) \in \mathbb{R}^2 : x^2 + (y+1)^2 > \frac{1}{4}\}$



ii)





iii) $x = k$ con $k < -\frac{1}{2}$ o $k > 2$ non intersecano mai l'insieme A.

$y = k$ con $k > \frac{1}{2}$ non intersecano mai l'insieme B.

2) i) $\mathcal{A}$ = "Pioverà nei prossimi giorni"

$\mathcal{B}$ = "il verde pubblico non subirà gravi danni".

Allora la proposizione data in matematichese si scrive "$\mathcal{A} \Rightarrow \mathcal{B}$".

Poichè $\mathcal{A} \Rightarrow \mathcal{B} \iff (\text{non } \mathcal{A}) \text{ o } \mathcal{B}$, allora non $[\mathcal{A} \Rightarrow \mathcal{B}] \Leftrightarrow$ non $[(\text{non } \mathcal{A}) \text{ o } \mathcal{B}] \Leftrightarrow [\text{non}(\text{non } \mathcal{A})]$ e $(\text{non } \mathcal{B})$ $\Leftrightarrow \mathcal{A}$ e $(\text{non } \mathcal{B})$. La negazione della proposizione data si scrive quindi come segue: "Pioverà nei prossimi giorni e il verde pubblico subirà gravi danni". ☐

ii) Vedi FILA (A), Es. 1) ii). ■

3) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = (x^2-2)^2$ $\qquad$ f è pari!

i) $f(x) \geq 0 \ \forall x \in \mathbb{R}$ e $= 0 \Leftrightarrow x = \pm\sqrt{2}$ $\qquad \lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty$

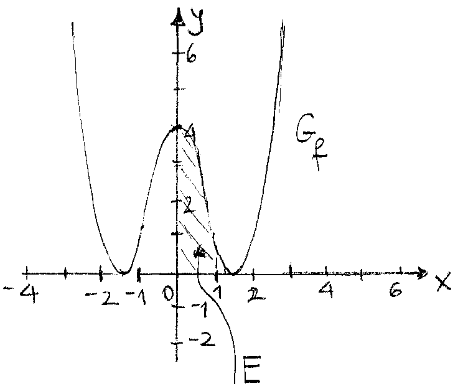
$f'(x) = 4x(x^2-2)$

	$-\sqrt{2}$		0		$\sqrt{2}$		
$-$		$+$		$-$		$+$	$f'(x)$
↘		↗		↘		↗	$f(x)$

$$f''(x) = 4(x^2-2) + 8x^2 = 12x^2 - 8 = 4(3x^2-2) \qquad f''(x) = 0 \Leftrightarrow x = \pm\sqrt{\tfrac{2}{3}}$$

	$-\sqrt{\frac{2}{3}}$		$\sqrt{\frac{2}{3}}$		
$+$		$-$		$+$	$f''(x)$
∪		∩		∪	$f(x)$

☐



ii) $f(\mathbb{R}) = [0, +\infty[$ ☐

iii) $\min_{\mathbb{R}} f = 0$ $\qquad x = -\sqrt{2}$, $x = \sqrt{2}$ pt. di minimo

$\nexists \max_{\mathbb{R}} f$. ☐

iv) $]-\infty, -\sqrt{2}]$: su questo intervallo non limitato inferiormente, f risulta iniettiva. ☐

v) $$\int_{-1}^{0} \frac{f'(x)}{f(x)}\,dx = \Big[\log|f(x)|\Big]_{-1}^{0} = \Big[\log(x^2-2)^2\Big]_{-1}^{0} = \log 4 - \cancel{\log 1} = \underline{\log 4}.$$

$$\int_0^1 f(x)\,dx = \int_0^1 (x^2-2)^2\,dx = \int_0^1 (x^4 - 4x^2 + 4)\,dx = \left[\frac{x^5}{5} - 4\frac{x^3}{3} + 4x\right]_0^1$$

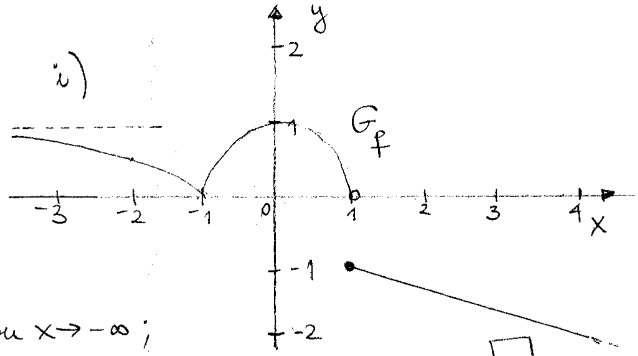
$$= \frac{1}{5} - \frac{4}{3} + 4 = \frac{3 - 20 + 60}{15} = \underline{\underline{\frac{43}{15}}}.$$

$\int_0^1 f(x)\,dx =$ area E. ■

4) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -2^{x+1}+1 & \text{se } x<-1 \\ -x^2+1 & \text{se } -1 \leq x < 1 \\ -\frac{x}{3} - \frac{2}{3} & \text{se } x \geq 1. \end{cases}$$

i)

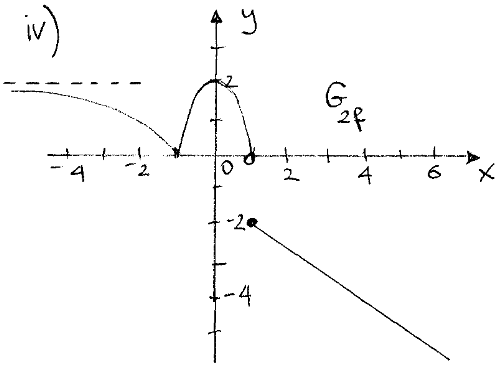


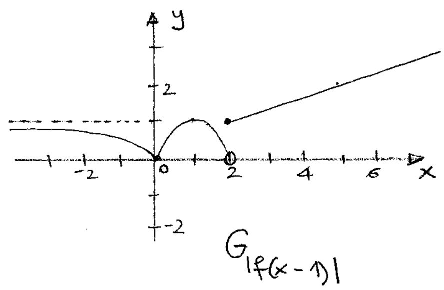
ii) $y = 1$ asintoto orizzontale per f, per $x \to -\infty$;

$y = -\frac{x}{3} - \frac{2}{3}$ asintoto obliquo per f, per $x \to +\infty$.

iii) $$\sum_{k=0}^{4} f(-2+k) = f(-2) + f(-1) + f(0) + f(1) + f(2) =$$
$$= \left(-\frac{1}{2}+1\right) + 0 + \cancel{1} - \cancel{1} - \frac{4}{3} = \underline{\underline{-\frac{5}{6}}}$$

iv)

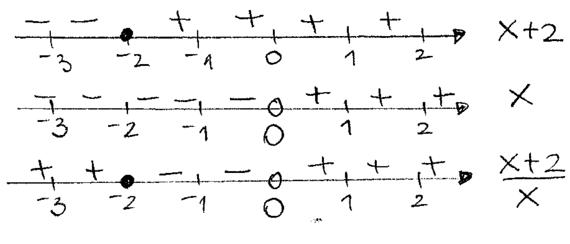




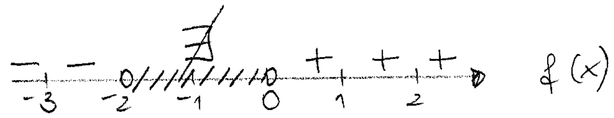
v) $(f \circ f)(0) = f(f(0)) = f(1) = -1$

è ben definito, poiché è definito $f(0)$ e $f(1)$.

5) $f(x) = \log_3\left(\frac{x+2}{x}\right)$ i) • $\text{dom} f = \left\{x \in \mathbb{R} : \frac{x+2}{x} > 0\right\} = \underline{\underline{]-\infty, -2[\cup]0, +\infty[}}$



• segno di f: $f(x) > 0 \iff \frac{x+2}{x} > 1 \iff \frac{2}{x} > 0 \Rightarrow x > 0$



- $\lim_{x\to-\infty} \log_3\left(\underbrace{\frac{x+2}{x}}_{\to 1}\right) = \underline{\underline{0}}$ $y=0$ asintoto orizzontale per f (per $x\to-\infty$).

  $\lim_{x\to-2^-} \log_3\left(\underbrace{\frac{x+2}{x}}_{\text{infinitesimo positivo}}\right) = \underline{\underline{-\infty}}$ $x=-2$ asintoto verticale per f

  $\lim_{x\to0^+} \log_3\left(\underbrace{\frac{x+2}{x}}_{\to+\infty}\right) = \underline{\underline{+\infty}}$ $x=0$ asintoto verticale per f

  $\lim_{x\to+\infty} \log_3\left(\underbrace{\frac{x+2}{x}}_{\to 1}\right) = \underline{\underline{0}}$ $y=0$ asintoto orizzontale per f (per $x\to+\infty$).

- f è continua, essendo rapporto di funzioni continue e composizione di funzioni continue.

- dom f' = dom f $f'(x)=\frac{1}{\left(\frac{x+2}{x}\right)\log 3}\cdot\left(\frac{x+2}{x}\right)' = \frac{\cancel{x}}{(\log 3)(x+2)}\cdot\frac{x-(x+2)}{x^{\cancel{2}}}$

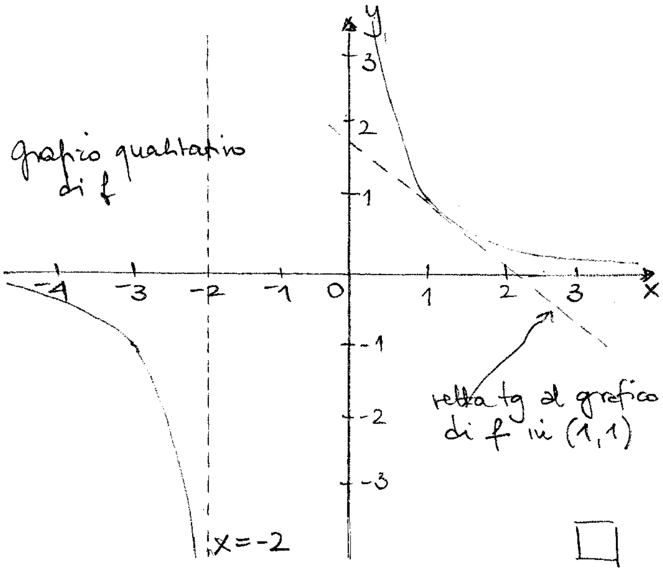
  $= \frac{-2}{(x^2+2x)\log 3}$

  $f'(x)$: $-$ (-3) $-$ (-2) ∄ (-1, 0) $-$ (1)
  $f(x)$: ↘ ↘

  ∄ sono pt. critici

- dom f'' = dom f $f''(x)=\frac{-2}{\log 3}\cdot\frac{-(2x+2)}{(x^2+2x)^2} = \frac{2}{\log 3}\cdot\frac{(2x+2)}{(x^2+2x)^2}$

  f'': $-$ (-3) $-$ (-2) ∄ (-1, 0) $+$ (1) $+$
  f: ∩ ∪



ii) $y = 1-\frac{2}{3\log 3}(x-1)$

$\Rightarrow$ $\underline{\underline{y = -\frac{2}{3\log 3}x + 1 + \frac{2}{3\log 3}}}$

6) $P_6 \cdot P_5 = \underline{\underline{6!\cdot 5!}}$