Università degli Studi di Trento – Facoltà di Scienze Cognitive
Corso di Laurea in Scienze e Tecniche di Psicologia Cognitiva
Corso di Laurea in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
ESAME SCRITTO DI ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2010 – 2011 – Rovereto, 14 febbraio 2011

FILA (A)

1) i) Sia $A(x,y)$ = " Nel Corso di Laurea x della Facoltà di SC è iscritto lo studente straniero y ".

Quindi la proposizione data si scrive : " $\forall x, \exists y : A(x,y)$ ".

La sua negazione è " $\exists x : \forall y$, non $A(x,y)$ ". $\square$

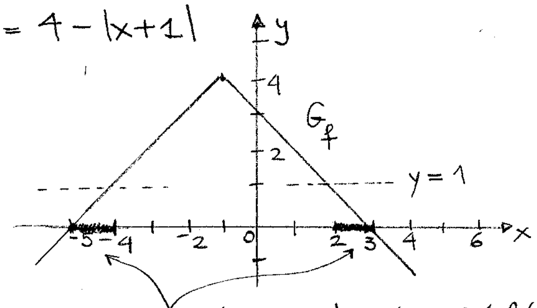
ii) a) $A = \{x \in \mathbb{R} : \ 0 \le 4 - |x+1| \le 1\}$

$$0 \le 4 - |x+1| \le 1$$
$$\iff 0 \ge -4 + |x+1| \ge -1$$
$$\iff 4 \ge |x+1| \ge 3$$

Ora
$$\begin{cases} 3 \le |x+1| \\ |x+1| \le 4 \end{cases} \iff \begin{cases} x+1 \le -3 \ \text{ o } \ x+1 \ge 3 \\ -4 \le x+1 \le 4 \end{cases} \iff \begin{cases} x \le -4 \ \text{ o } \ x \ge 2 \\ -5 \le x \le 3 \end{cases}$$

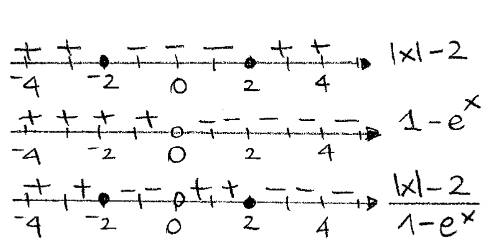
$$\Rightarrow \quad \underline{\underline{A = [-5,-4] \cup [2,3]}}.$$

NOTA: si può determinare rapidamente l'insieme A "leggendo" bene il grafico della funzione $f(x) = 4 - |x+1|$



per queste x si ha che $0 \le f(x) \le 1$

$$B = \left\{x \in \mathbb{R} : \frac{|x|-2}{1-e^x} > 0\right\}$$
$$= \underline{\underline{]-\infty, -2[\ \cup \ ]0, 2[}}.$$



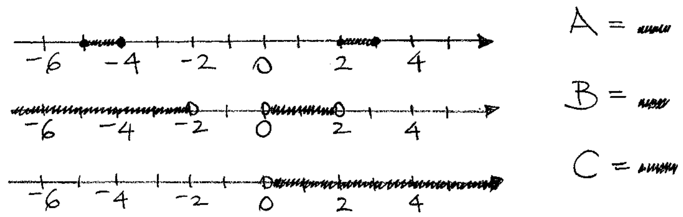
$C = \{x \in \mathbb{R} : \log_2(3x) - \log_2(x^2+1) < 1\}$

$$\log_2(3x) - \log_2(x^2+1) < 1 \iff \begin{cases} 3x > 0 \\ x^2+1 > 0 \\ \log_2 \dfrac{3x}{x^2+1} < \log_2 2 \end{cases}$$

$$\iff \begin{cases} x > 0 \\ \dfrac{3x}{x^2+1} < 2 \end{cases}$$ poiché $x^2+1>0 \ \forall x \in \mathbb{R}$ e $\log_2 x$ è una funz. crescente.

$$\iff \begin{cases} x > 0 \\ \dfrac{3x-2x^2-2}{x^2+1} < 0 \end{cases} \iff \begin{cases} x > 0 \\ \dfrac{2x^2-3x+2}{x^2+1} > 0 \end{cases}.$$

Poiché $2x^2-3x+2>0 \ \forall x \in \mathbb{R}$ si conclude $\underline{\underline{C =]0,+\infty[}}$.



b) $A \cap B = \underline{\underline{[-5,-4]}}$; $\qquad C \cup B = \underline{\underline{]-\infty,-2[\cup]0,+\infty[}}$.

2) $y + x^2 + 2x - 2 = 0 \iff y = -x^2-2x+2$

$= -(x^2+2x) + 2$

$= -(x+1)^2 + 1 + 2 \quad \Rightarrow y = -(x+1)^2 + 3$

$\underline{\underline{V = (-1,3)}}$

$x^2+y^2-2x-8y+16=0 \iff x^2-2x+y^2-8y+16=0$

$(x-1)^2-1+(y-4)^2-\cancel{16}+\cancel{16}=0$

$(x-1)^2+(y-4)^2=1$

$\underline{\underline{C_1=(1,4)}} \quad \underline{\underline{r=1}}$

$x^2+4y^2-4x-8y+4=0 \iff x^2-4x+4(y^2-2y)+4=0$

$(x-2)^2-4+4(y-1)^2-\cancel{4}+\cancel{4}=0$

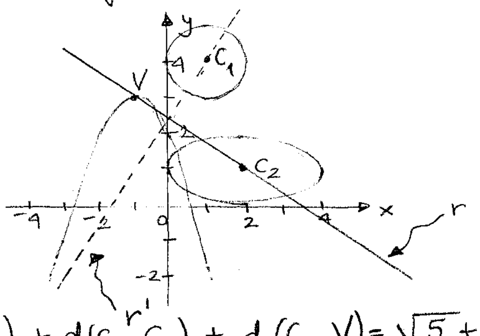
$(x-2)^2+4(y-1)^2=4$

$\dfrac{(x-2)^2}{4}+(y-1)^2=1 \quad \underline{\underline{C_2=(2,1)}} \quad \underline{\underline{a=2,\ b=1}}.$

i) retta r : $\dfrac{y-3}{1-3} = \dfrac{x+1}{2+1} \Rightarrow \underline{\underline{y = -\frac{2}{3}x + \frac{7}{3}}}$.

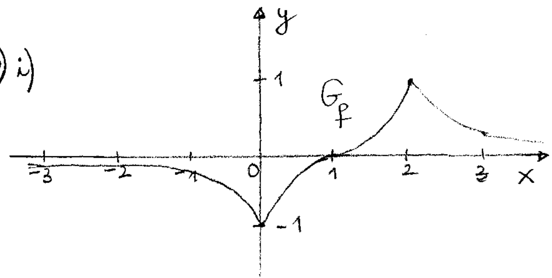
ii) retta r' : $y = 4 + \frac{3}{2}(x-1) \Rightarrow \underline{\underline{y = \frac{3}{2}x + \frac{5}{2}}}$.

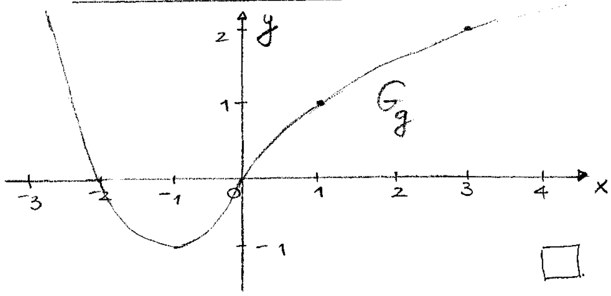
iii)



iv) $P = d(V, C_1) + d(C_1, C_2) + d(C_2, V) = \underline{\underline{\sqrt{5} + \sqrt{10} + \sqrt{13}}}$.

3) i)





ii) $f(\mathbb{R}) = \underline{\underline{[-1, 1]}}$.

iii) $\min_{\mathbb{R}} f = -1$ $x = 0$ pt. di minimo ;

$\max_{\mathbb{R}} f = 1$ $x = 2$ pt. di massimo .

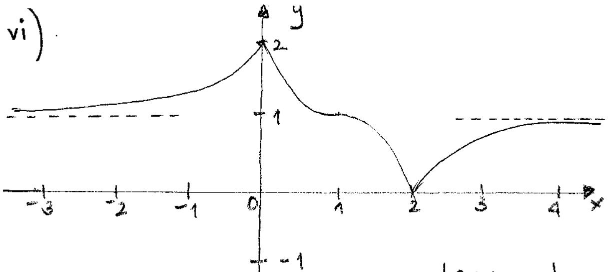
iv) $\underline{\underline{[-1, +\infty[}}$.

v) $(f+g)(0) = f(0) + g(0) = -1 + 0 = \underline{\underline{-1}}$;

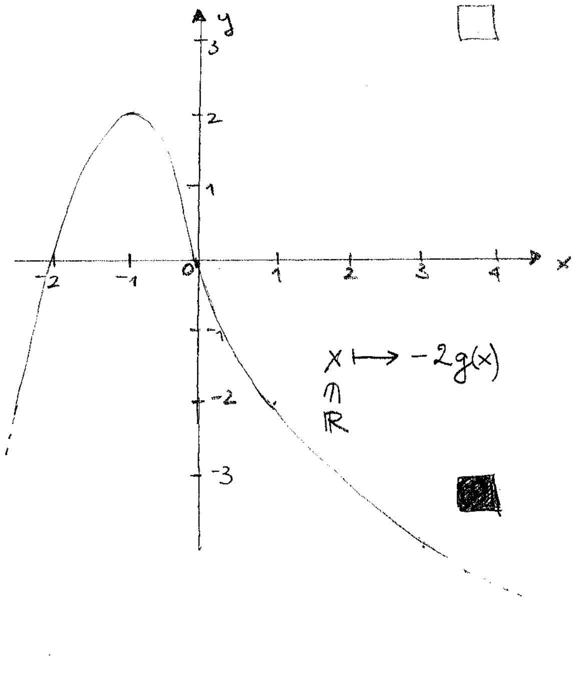
$(f \circ g)(3) = f(g(3)) = f(2) = \underline{\underline{1}}$;

$(g \circ f)(0) = g(f(0)) = g(-1) = \underline{\underline{-1}}$.

vi)

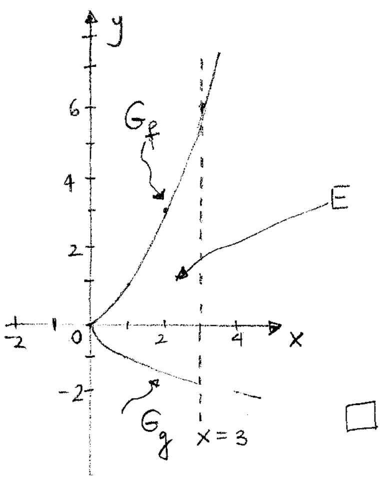


$\underset{\mathbb{R}}{\overset{\cap}{x}} \longmapsto |f(x) - 1|$



$\underset{\mathbb{R}}{\overset{\cap}{x}} \longmapsto -2g(x)$

4) i)



ii) $\sum_{n=1}^{4} (-1)^n f(n) = (-1)^1(2^1-1) + (-1)^2(2^2-1) + (-1)^3(2^3-1) + (-1)^4(2^4-1) =$

$= -1 + 3 - 7 + 15 = \underline{\underline{10}}$

$\sum_{n=1}^{4} g'(n^2) = -\frac{1}{2} - \frac{1}{2\cdot 2} - \frac{1}{2\cdot 3} - \frac{1}{2\cdot 4} =$

$\left[g'(x) = -\frac{1}{2\sqrt{x}} \Rightarrow g'(n^2) = -\frac{1}{2n}\right]$

$= -\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \underline{\underline{-\frac{25}{24}}}.$

iii) $\text{area}\, E = \int_0^3 [f(x) - g(x)]\,dx = \int_0^3 \left[2^x - 1 + \sqrt{x}\right]dx = \left[\frac{2^x}{\log 2} - x + \frac{x^{3/2}}{3/2}\right]_0^3 =$

$= \frac{2^3}{\log 2} - 3 + \frac{2}{3}\cdot 3^{3/2} - \frac{1}{\log 2} = \underline{\underline{\frac{7}{\log 2} - 3 + 2\sqrt{3}}}.$

5) i) $f(x) = \frac{x^2+5}{x-2}$

- $\text{dom} f = \mathbb{R}\setminus\{2\} =]-\infty, 2[\, \cup\,]2, +\infty[$.
- segno di f

$$\begin{array}{c|c|c} x & <2 & >2 \\ \hline f & - & + \end{array}$$

(NOTA: $x^2+5 > 0 \;\; \forall x \in \mathbb{R}$)

- $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to+\infty} f(x) = +\infty$ } f non ha asintoti orizzontali!

$\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = +\infty$ — $x=2$ è asintoto vert.

Osserviamo che $\lim_{\substack{x\to-\infty \\ (+\infty)}} \left[\frac{f(x)}{x}\right] = \lim_{\substack{x\to-\infty \\ (+\infty)}} \frac{x^2+5}{x^2-2x} = 1$

$\lim_{\substack{x\to-\infty \\ (+\infty)}} [f(x) - x] = \lim_{\substack{x\to-\infty \\ (+\infty)}} \left[\frac{\cancel{x^2}+5-\cancel{x^2}+2x}{x-2}\right] = 2$

$\Rightarrow y = x+2$ è asintoto obliquo per f per $x \to -\infty\ (+\infty)$.

- f è continua su $\mathbb{R}\setminus\{2\}$, poiché rapporto di funzioni continue sul dom f.
- $\text{dom} f' = \mathbb{R}\setminus\{2\}$ $\quad f'(x) = \frac{2x\cdot(x-2) - (x^2+5)\cdot 1}{(x-2)^2} = \frac{x^2-4x-5}{(x-2)^2}$

$f'(x) = 0 \Leftrightarrow x = -1,\ x = 5$ pt. critici di f;

$f(-1) = -2$

$f(5) = 10.$

$$\begin{array}{c|c|c|c|c|c} x & <-1 & -1<x<2 & 2 & 2<x<5 & >5 \\ \hline x^2-4x-5 & + & - & & - & + \\ (x-2)^2 & + & + & 0 & + & + \\ f'(x) & + & - & & - & + \\ f(x) & \nearrow & \searrow & & \searrow & \nearrow \end{array}$$

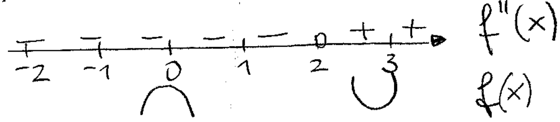
$x=-1$ pt. di max loc. stretto per f

$x=5$ pt. di min. loc. stretto per f

• dom $f'' = \mathbb{R}\setminus\{2\}$

$$f''(x) = \frac{(2x-4)(x-2)^{\cancel{2}} - (x^2-4x-5)\cdot 2\cancel{(x-2)}}{(x-2)^{\cancel{4}\,3}} =$$

$$= \frac{\cancel{2x^2} - \cancel{8x} + 8 - \cancel{2x^2} + \cancel{8x} + 10}{(x-2)^3} = \frac{18}{(x-2)^3}.$$



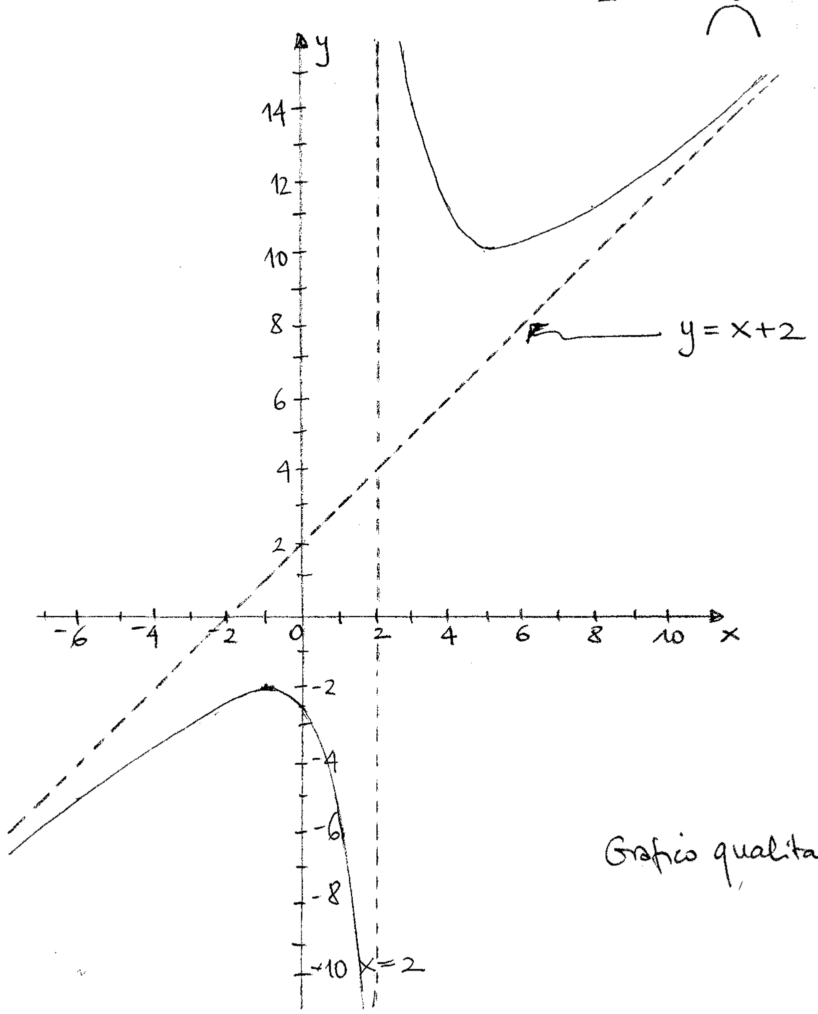


Grafico qualitativo di f

ii) $x+2+\frac{9}{x-2} = \frac{(x+2)(x-2)+9}{x-2} = \frac{x^2-4+9}{x-2} = \frac{x^2+5}{x-2} = f(x)\ \forall x \in \mathbb{R}\setminus\{2\}.$

iii) $\int_0^1 f(x)\,dx \underset{ii)}{=} \int_0^1 \left(x+2+\frac{9}{x-2}\right)dx = \left[\frac{x^2}{2}+2x+9\log|x-2|\right]_0^1 = \left(\frac{1}{2}+2\right) - 9\log 2 = \underline{\underline{\frac{5}{2} - 9\log 2}}.$

6) i) Non sono soddisfatte le ipotesi del teorema di Weierstrass, poiché f non è continua in $x=0$ $\left(f(0)=0 \neq \lim_{x\to 0^+} f(x) = 1\right)$. Non è verificata nemmeno la tesi poiché $\nexists \max_{[0,2]} f$ $\left(\exists \min_{[0,2]} f = -2\right)$.

ii) Non sono soddisfatte le ipotesi del teorema di esistenza degli zeri: f non è continua in $[0,2]$ (vedi i)), e

$f(0)=0$, $f(2)=-2$.

È verificata la tesi del teorema: $\exists\, x_0 \in\,]0,2[$ t.c. $f(x_0)=0$; infatti $f(1)=0$. ■

FILA B

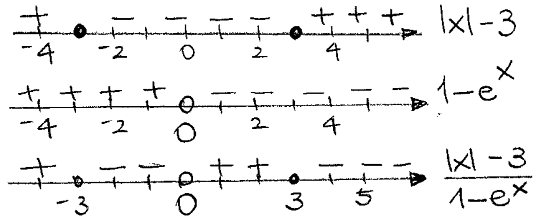
1) i) Sia $A(x,y)=$ "Nel Corso di laurea x dell'Università di Bologna è iscritto lo studente straniero y".

Quindi la proposizione data si scrive: "$\exists x: \forall y,\ A(x,y)$".

La sua negazione è "$\forall x, \exists y:$ non $A(x,y)$". □

ii) a) $A=\left\{x\in\mathbb{R}: \dfrac{|x|-3}{1-e^x}>0\right\}$

$= \underline{]-\infty,-3[\ \cup\ ]0,3[}$.



$B=\{x\in\mathbb{R}: 0\le 3-|x+1|\le 1\}$

$0\le 3-|x+1|\le 1$

$\iff 0\ge -3+|x+1|\ge -1$

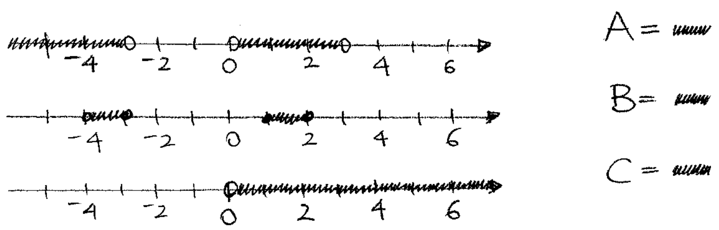
$\iff 3\ge |x+1|\ge 2$

Ora $\begin{cases} 2\le |x+1| \\ |x+1|\le 3\end{cases} \iff \begin{cases} x+1\le -2 \text{ o } x+1\ge 2 \\ -3\le x+1\le 3\end{cases} \iff \begin{cases} x\le -3 \text{ o } x\ge 1 \\ -4\le x\le 2\end{cases}$

$\Rightarrow \underline{B=[-4,-3]\cup[1,2]}$. (vedi anche FILA A, Es. 1 ii) NOTA)

$\underline{C=\,]0,+\infty[}$ (vedi FILA A, Es. 1 ii)) Oss. che in questo caso abbiamo $\log_2(3x) - \log_2(x^2+1) \leq 1 \iff \begin{cases} x > 0 \\ \dfrac{2x^2-3x+2}{x^2+1} \geq 0 \end{cases}$

Poiché $2x^2-3x+2>0 \ \forall x \in \mathbb{R}$, si ha quindi che la 2^a diseq. è sempre verificata e quindi si ha $C =]0,+\infty[$.



b) $A \cap B = \underline{\underline{[-4,-3[\cup [1,2]}}$ $\qquad C \cup B = \underline{\underline{[-4,-3] \cup]0,+\infty[}}$.

2) $y + x^2 - 2x - 2 = 0 \iff y = -x^2 + 2x + 2$

$$= -(x^2-2x) + 2$$

$$= -(x-1)^2 + 1 + 2 \Rightarrow y = -(x-1)^2 + 3$$

$$\underline{\underline{V = (1,3)}}$$

$x^2 + y^2 - 6x + 4y + 12 = 0 \iff x^2 - 6x + y^2 + 4y + 12 = 0$

$$(x-3)^2 - 9 + (y+2)^2 - 4 + 12 = 0$$

$$(x-3)^2 + (y+2)^2 = 1$$

$$\underline{\underline{C_1 = (3,-2)}} \quad \underline{\underline{r = 1}}$$

$x^2 + 4y^2 + 4x - 8y + 4 = 0 \iff x^2 + 4x + 4y^2 - 8y + 4 = 0$

$$(x+2)^2 - 4 + 4(y-1)^2 \cancel{-4} + \cancel{4} = 0$$

$$(x+2)^2 + 4(y-1)^2 = 4$$

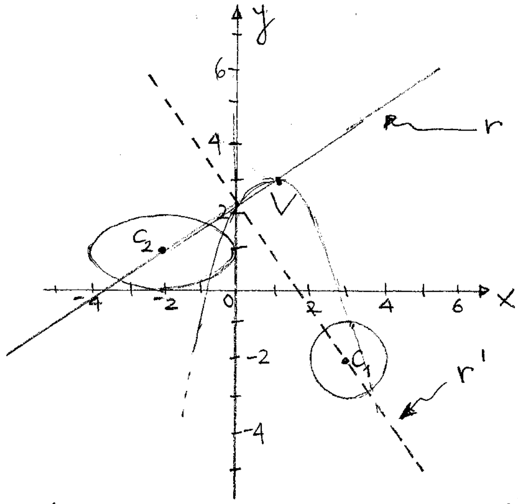
$$\frac{(x+2)^2}{4} + (y-1)^2 = 1$$

$$\underline{\underline{C_2 = (-2,1)}} \quad \underline{\underline{a = 2,\ b = 1.}}$$

i) retta r : $\dfrac{y-3}{1-3} = \dfrac{x-1}{-2-1} \Rightarrow \underline{\underline{y = \frac{2}{3}x + \frac{7}{3}}}$.

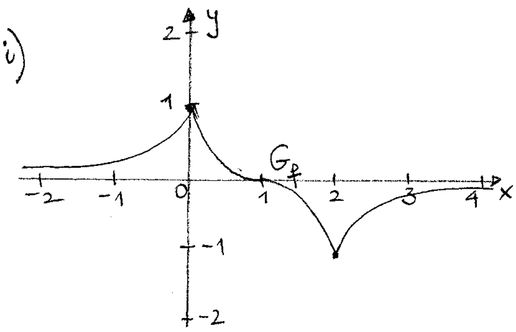
ii) retta r' : $y = -2 - \frac{3}{2}(x-3) \Rightarrow \underline{\underline{y = -\frac{3}{2}x + \frac{5}{2}}}$.

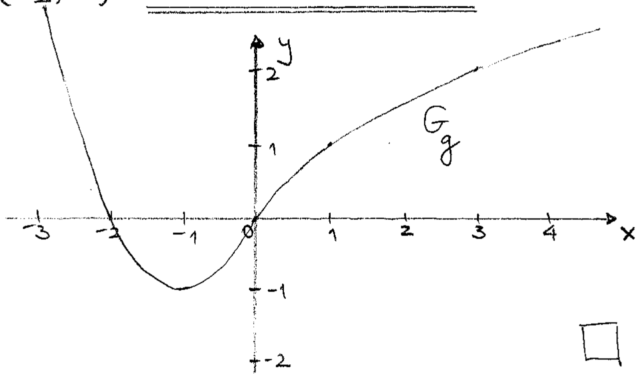
iii)



iv) $P = d(V,C_1) + d(C_1,C_2) + d(C_2,V) = \underline{\underline{\sqrt{29} + \sqrt{34} + \sqrt{13}}}$.

3) i)





ii) $f(\mathbb{R}) = [-1,1]$.

iii) $\min_{\mathbb{R}} f = -1 \qquad x=2$ pt. di minimo ;

$\max_{\mathbb{R}} f = 1 \qquad x=0$ pt. di massimo .

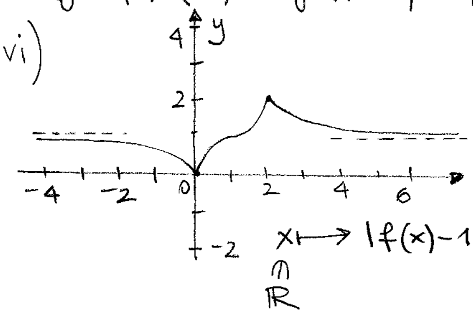
iv) $[-1,+\infty[$.

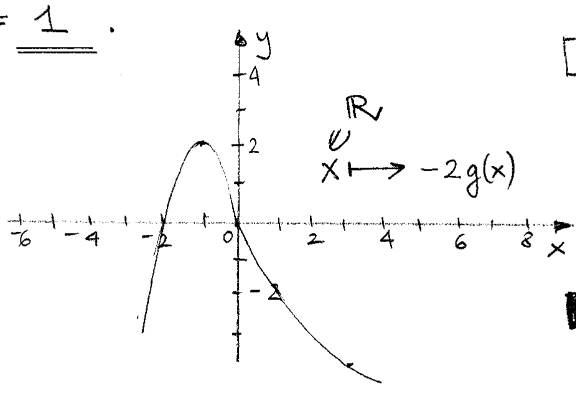
v) $(f+g)(0) = f(0) + g(0) = 1 + 0 = \underline{\underline{1}}$;

$(f \circ g)(3) = f(g(3)) = f(2) = \underline{\underline{-1}}$;

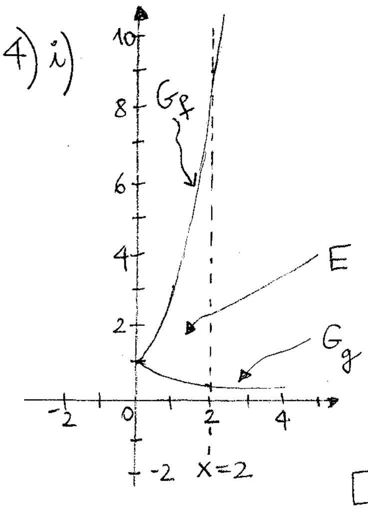
$(g \circ f)(0) = g(f(0)) = g(1) = \underline{\underline{1}}$.

vi)





4) i)



$\square$

ii) $\displaystyle\sum_{n=0}^{4} (-1)^{n+1} f(n) = (-1)^1 \cdot 3^0 + (-1)^2 \cdot 3^1 + (-1)^3 \cdot 3^2 + (-1)^4 \cdot 3^3 + (-1)^5 \cdot 3^4 =$

$= -1 + 3 - 9 + 27 - 81 = \underline{\underline{-61}}.$

$\displaystyle\sum_{n=0}^{3} g'(n) = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} = \underline{\underline{-\frac{205}{144}}}$

$\left[g'(x) = -\dfrac{1}{(x+1)^2} \Rightarrow g'(n) = \dfrac{-1}{(n+1)^2} \right]$

$\square$

iii) $\text{area } E = \displaystyle\int_0^2 \left[f(x) - g(x) \right] dx = \int_0^2 \left[3^x - \frac{1}{x+1} \right] dx = \left[\frac{3^x}{\log 3} - \log|x+1| \right]_0^2$

$= \left(\dfrac{9}{\log 3} - \log 3 \right) - \dfrac{1}{\log 3} = \underline{\underline{\dfrac{8}{\log 3} - \log 3.}}$

$\blacksquare$

5) i) $f(x) = \dfrac{x^2+7}{x-3}$

- $\operatorname{dom} f = \mathbb{R} \setminus \{3\} =]-\infty, 3[\cup]3, +\infty[$.
- segno di f: $-$ per $x < 3$, $+$ per $x > 3$ (retta: -2, 0, 3, 5, 7)

(NOTA: $x^2 + 7 > 0 \ \forall x \in \mathbb{R}$)

- $\displaystyle\lim_{x \to -\infty} f(x) = -\infty$, $\displaystyle\lim_{x \to +\infty} f(x) = +\infty$ } f non ha asintoti orizzontali!

$\displaystyle\lim_{x \to 3^-} f(x) = -\infty$, $\displaystyle\lim_{x \to 3^+} f(x) = +\infty$ } $x = 3$ è asintoto verticale

Osserviamo che $\displaystyle\lim_{\substack{x \to -\infty \\ (+\infty)}} \left[\frac{f(x)}{x} \right] = \lim_{\substack{x \to -\infty \\ (+\infty)}} \frac{x^2+7}{x^2-3x} = 1$

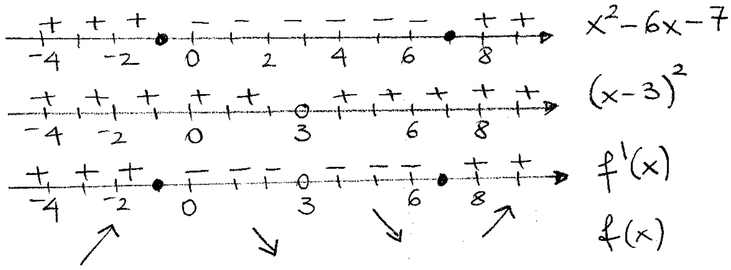
$\displaystyle\lim_{\substack{x \to -\infty \\ (+\infty)}} \left[f(x) - x \right] = \lim_{\substack{x \to -\infty \\ (+\infty)}} \left[\frac{\cancel{x^2} + 7 - \cancel{x^2} + 3x}{x-3} \right] = 3$

} $\Rightarrow y = x + 3$ è asintoto obliquo per f, per $x \to -\infty$ $(+\infty)$.

- f è continua su $\mathbb{R} \setminus \{3\}$, poiché rapporto di funzioni continue sul dom f.
- $\operatorname{dom} f' = \mathbb{R} \setminus \{3\}$ $\quad f'(x) = \dfrac{2x(x-3) - (x^2+7) \cdot 1}{(x-3)^2} = \dfrac{x^2 - 6x - 7}{(x-3)^2}$

$f'(x) = 0 \iff x = -1, \ x = 7$ pt. critici di f;

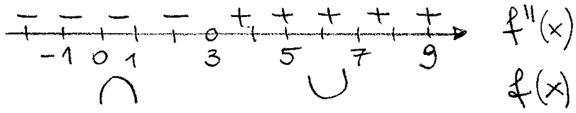
$f(-1) = -2 \qquad f(7) = 14.$

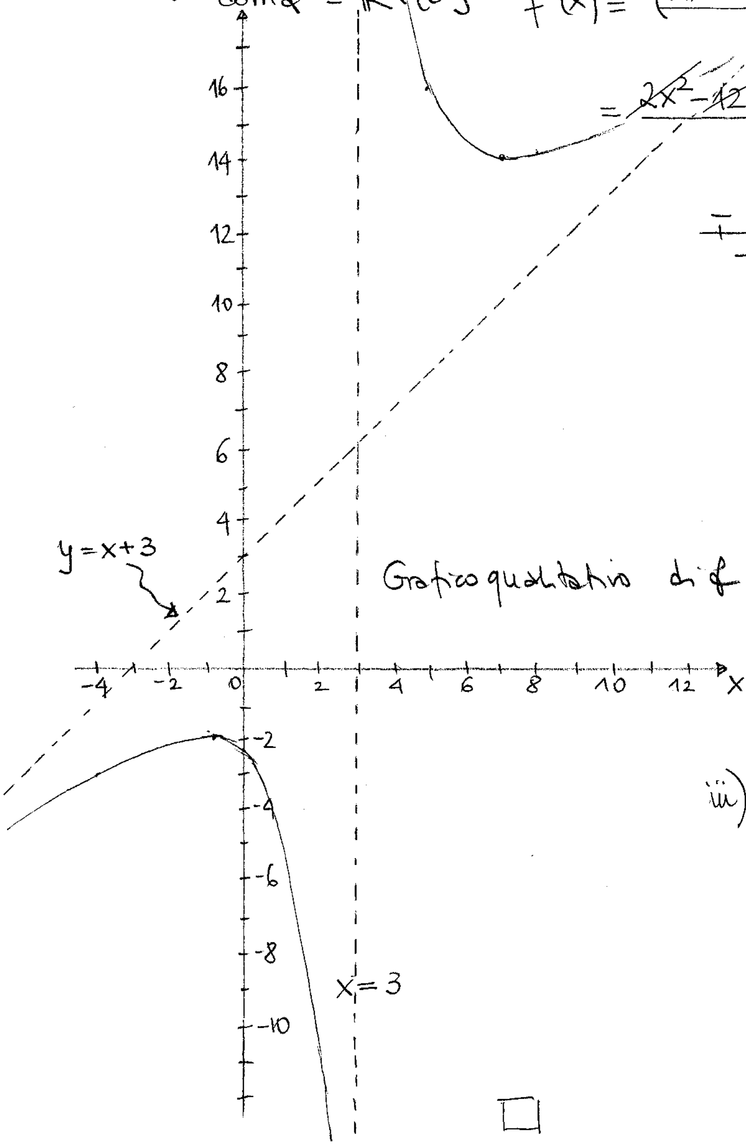


$x=-1$ pt. di max. loc. stretto per f $\qquad$ $x=7$ pt di min. loc. stretto per f

• $\operatorname{dom} f'' = \mathbb{R}\setminus\{3\}$ $\quad f''(x) = \dfrac{(2x-6)(x-3)^{\cancel{2}}-(x^2-6x-7)\cdot 2\cancel{(x-3)}}{(x-3)^{\cancel{4}\,3}} =$

$$= \frac{\cancel{2x^2}-\cancel{12x}+18-\cancel{2x^2}+\cancel{12x}+14}{(x-3)^3} = \frac{32}{(x-3)^3}$$





ii) $x+3+\dfrac{16}{x-3} = \dfrac{(x+3)(x-3)+16}{x-3} =$

$$= \frac{x^2-9+16}{x-3} = \frac{x^2+7}{x-3} = f(x) \quad \forall x\in\mathbb{R}\setminus\{3\}$$

iii) $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 \left(x+3+\frac{16}{x-3}\right)dx =$

$$= \left[\frac{x^2}{2}+3x+16\log|x-3|\right]_0^1 =$$

$$= \left(\frac{1}{2}+3+16\log 2\right)-\left(16\log 3\right) =$$

$$= \frac{7}{2}+16\log\frac{2}{3}.$$

6) i) Non sono soddisfatte le ipotesi del teorema di Weierstrass, poiché f non è continua in $x=0$ $\left(f(0)=1 \neq \lim_{x\to 0^-} f(x)=0\right)$. È invece verificata la tesi del teorema: $\exists \min_{[0,2]} f = f(1)$, $\exists \max_{[0,2]} f = f(0)=1$.

ii) f non è continua in $x=0$! Quindi non sono soddisfatte tutte le ipotesi del teorema di Zeri degli zeri. La tesi però è soddisfatta: $\exists x_0 \in]0,2[: f(x_0)=0$.