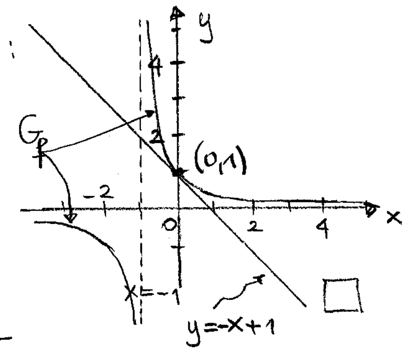
Università degli Studi di Trento - Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione - CdL in Filosofia
Corso di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2010-2011 - Rovereto, 29 novembre - 3 dicembre 2010

1) i) $f(x) = \frac{1}{x+1}$ $\quad f'(x) = -\frac{1}{(x+1)^2}$ $\quad f'(0) = -1$

Eq. retta tg. al grafico di f nel pt. $(0,1)$:

$y = 1 - 1(x-0) \Rightarrow \underline{y = -x+1}$



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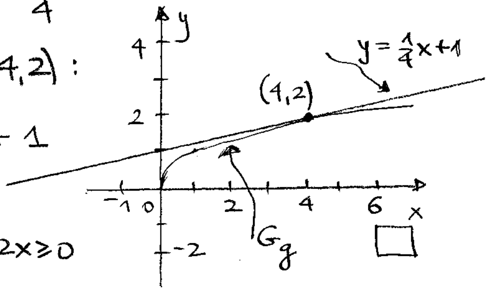
ii) $g(x) = \sqrt{x}$ $\quad g'(x) = \frac{1}{2\sqrt{x}}$ se $x > 0$

$g'(4) = \frac{1}{4}$

Eq. retta tg. al grafico di g nel pt. $(4,2)$:

$y = 2 + \frac{1}{4}(x-4) \Rightarrow y = \frac{1}{4}x + 1$



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iii) $h(x) = |x^2+2x|$ $\quad h(x) = \begin{cases} x^2+2x & \text{se } x^2+2x \geq 0 \\ -(x^2+2x) & \text{se } x^2+2x < 0 \end{cases}$

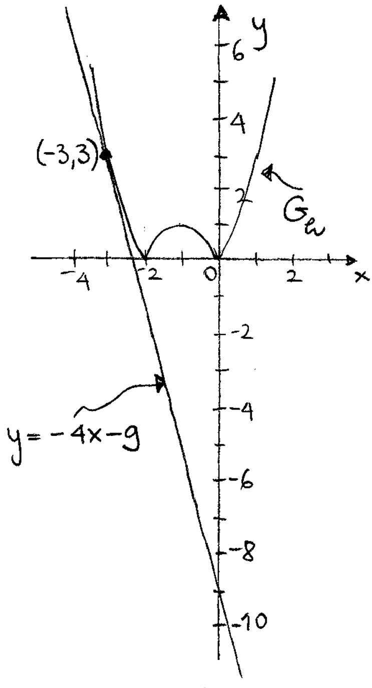
Ora $x^2+2x \geq 0 \Leftrightarrow x \in]-\infty, -2] \cup [0, +\infty[$.

Ora $x = -3$ è contenuto all'interno dell'insieme in cui $x^2+2x \geq 0$ e quindi $h(x) = x^2+2x$ in un intorno di $x=-3$.

Abbiamo allora $h'(x) = 2x+2$ e $h'(-3) = -4$.

Eq. retta tg. al grafico di h nel pt. $(-3,3)$:

$y = 3 - 4(x+3) \Rightarrow y = -4x-9$



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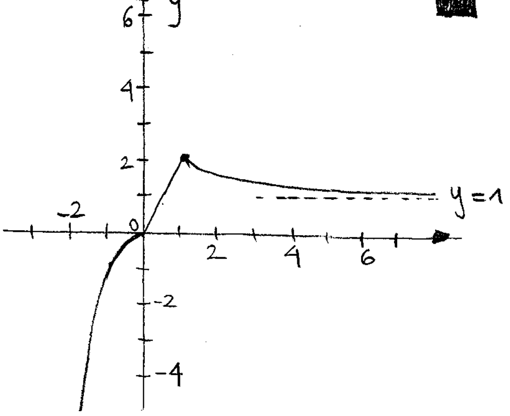
2) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} x^3 & \text{se } x \leq 0 \\ 2x & \text{se } 0 < x \leq 1 \\ \frac{1}{x}+1 & \text{se } x > 1 \end{cases}$



i) $$\left.\begin{array}{l}\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} x^3 = 0 = f(0)\\ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 2x = 0\end{array}\right\} \Rightarrow \exists \lim_{x\to 0} f(x) = 0 = f(0).$$

$$\left.\begin{array}{l}\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 2x = 2 = f(1)\\ \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \left(\frac{1}{x}+1\right) = 2\end{array}\right\} \Rightarrow \exists \lim_{x\to 1} f(x) = 2 = f(1).$$ ☐

ii) $$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^3-0}{h} = \lim_{h\to 0^-} h^2 = \underline{\underline{0}} \quad \left(= \underline{\underline{f'_-(0)}}\right)$$

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{2h-0}{h} = \lim_{h\to 0^+} 2 = \underline{\underline{2}} \quad \left(= \underline{\underline{f'_+(0)}}\right)$$

Quindi $\nexists \lim\limits_{h\to 0} \frac{f(h)-f(0)}{h}$ e f non è derivabile in $x_0 = 0$. ☐

iii) $$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{2(1+h)-2}{h} = \lim_{h\to 0^-} \frac{2h}{h} = \underline{\underline{2}} \quad \left(\underline{\underline{= f'_-(1)}}\right)$$

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{\left(\frac{1}{h+1}+1\right)-2}{h} = \lim_{h\to 0^+} \frac{\frac{1}{h+1}-1}{h}$$

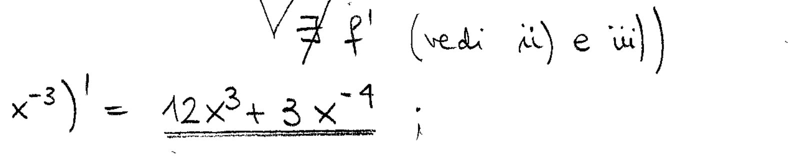
$$= \lim_{h\to 0^+} \frac{\cancel{1}-h-\cancel{1}}{h(h+1)} = \lim_{h\to 0} \frac{-1}{h+1} = \underline{\underline{-1}} \quad \left(\underline{\underline{= f'_+(1)}}\right)$$

Quindi $\nexists \lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h}$ ed f non è derivabile in $x_1 = 1$. ☐

iv) se $x<0$ si ha $f'(x) = 3x^2$ e quindi $f'(x) > 0 \quad \forall\, x<0$;

se $0<x<1$ si ha $f'(x) = 2$ e quindi $f'(x) > 0 \quad \forall\, 0<x<1$;

se $x>1$ si ha $f'(x) = -\frac{1}{x^2}$ e quindi $f'(x) < 0 \quad \forall\, x>1$.



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3) i) a) $$\left(3x^4 - x^{-3}\right)' = \underline{\underline{12x^3 + 3x^{-4}}};$$

$$\left(\frac{x+x^2}{x^3}\right)' = \left(\frac{1}{x^2}+\frac{1}{x}\right)' = \underline{\underline{\frac{-2}{x^3} - \frac{1}{x^2}}};$$

$$\left(\frac{x^{-2}}{4-x^2}\right)' = \frac{-2x^{-3}(4-x^2) - x^{-2}(-2x)}{(4-x^2)^2} = \underline{\underline{\frac{-8x^{-3}+4x^{-1}}{(4-x^2)^2}}};$$

b) $\left[(e^{-x}+x^2)\log_3 x\right]' = (-e^{-x}+2x)\log_3 x + \frac{(e^{-x}+x^2)}{x\log 3}$;

$\left[(x^{-\frac{1}{3}}+4^x)(\log x - x)\right]' = \left(-\frac{1}{3}x^{-\frac{4}{3}}+4^x\log 4\right)(\log x - x)+\left(x^{-\frac{1}{3}}+4^x\right)\left(\frac{1}{x}-1\right).$

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ii) $\left[(3x+e^{2x})^{-1}\right]' = -(3x+e^{2x})^{-2}(3+2e^{2x})$;

$\left(2^{x^2+1}\right)' = 2^{x^2+1}\cdot\log 2\cdot(2x)$;

$\left[\log(4x+3x^2)\right]' = \frac{4+6x}{4x+3x^2}$.

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4) i) $\lim_{x\to -1^-} f(x) = 1$; $\lim_{x\to -1^+} f(x) = 1$; $\lim_{x\to 1^-} f(x) = 2$; $\lim_{x\to 1^+} f(x) = 2$.

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ii) $x=-4$, $x=-1$, $x=1$ sono pt. di max. loc. per f

($x=-1$, $x=1$ sono, in particolare, pt. di massimo per f).

$x=-2$, $x=0$ sono pt. di min. loc. per f.

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iii) $(-4,1)$, $(-2,-2)$, $(0,0)$ sono i pt. nel grafico di f con retta tg. orizzontale.

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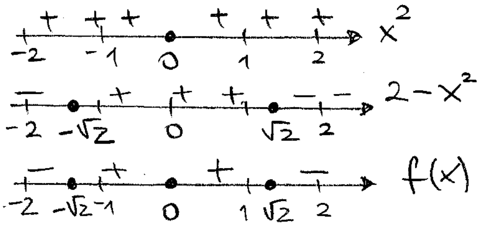
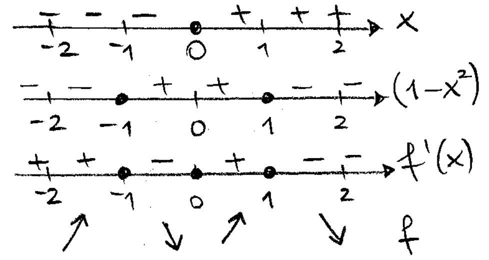
5) $\boxed{f(x)=4x^2-2x^4}$

- dom $f = \mathbb{R} =]-\infty,+\infty[$ f è una funzione pari!
- $f(x) = 2x^2(2-x^2)$



- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = -\infty$
- f è continua in tutto $\mathbb{R}$ essendo prodotto e somma di funzioni continue.
- dom $f' = \mathbb{R}$ $f'(x) = 8x-8x^3 = 8x(1-x^2)$

$f'(x)=0 \iff \left.\begin{array}{l} x=0 \\ x=\pm 1 \end{array}\right\}$ pt. critici per f



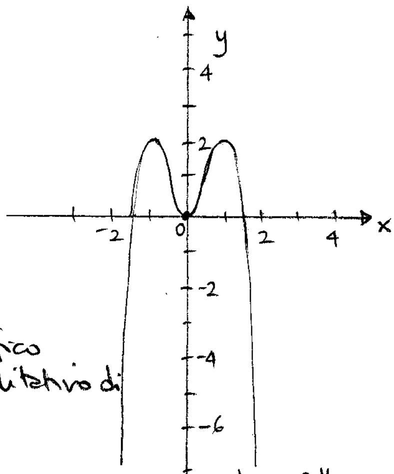
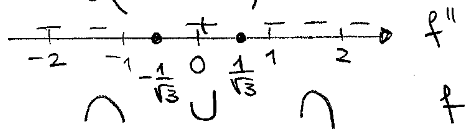


grafico qualitativo di f

$x=-1$ pt. di max loc. $f(-1)=2$
$x=1$ 〃 $f(1)=2$

$x=0$ pt. di min. loc. $f(0)=0$.

- dom $f'' = \mathbb{R}$ $f''(x) = 8-24x^2 = 8(1-3x^2)$



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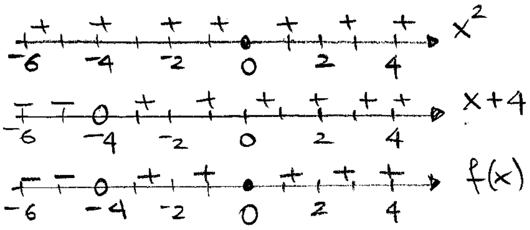
$$\boxed{f(x)=\frac{x^2}{x+4}}$$

- dom $f = \mathbb{R}\setminus\{-4\} =]-\infty,-4[\,\cup\,]-4,+\infty[$
- segno di f



- $\lim\limits_{x\to-\infty} f(x) = -\infty$ f non ha asint. orizz. per f, per $x\to-\infty$

$\lim\limits_{x\to-4^-} f(x) = -\infty$ $x=-4$ asint. verticale per f

$\lim\limits_{x\to-4^+} f(x) = +\infty$

$\lim\limits_{x\to+\infty} f(x) = +\infty$ f non ha asintoto orizz. per f, per $x\to+\infty$.

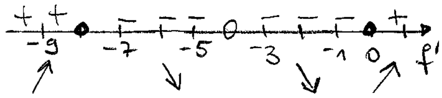
$$\left.\begin{array}{l} \lim\limits_{\substack{x\to-\infty\\(+\infty)}} \dfrac{f(x)}{x} = \lim\limits_{\substack{x\to-\infty\\(+\infty)}} \dfrac{x^2}{x^2+4x} = 1 = a \\ \lim\limits_{\substack{x\to-\infty\\(+\infty)}} \left[f(x)-x\right] = \lim\limits_{\substack{x\to-\infty\\(+\infty)}} \left[\dfrac{x^2}{x+4}-x\right] = \lim\limits_{\substack{x\to-\infty\\(+\infty)}} \dfrac{-4x}{x+4} = -4 = b \end{array}\right\} \Rightarrow$$

$y = x-4$ è asint. obliquo per f

per $x\to-\infty$ (e per $x\to+\infty$).

- f è continua in $\mathbb{R}\setminus\{-4\}$, essendo rapporto di funzioni continue e il denominatore si annulla solo in $x=-4$.
- $\mathrm{dom}\, f' = \mathbb{R}\setminus\{-4\}$ $\quad f'(x) = \frac{2x(x+4)-x^2}{(x+4)^2} = \frac{x^2+8x}{(x+4)^2} = $

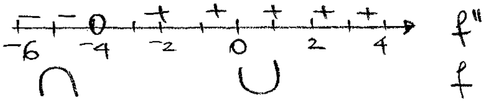
$$= \frac{x(x+8)}{(x+4)^2}$$



$x=0$ | $x=-8$ pr. critici per f e $\quad f(0)=0$, $f(-8)=-16$ $\quad x=-8$ pr. di max. loc., $x=0$ pr. di min. loc.

- dor $= \mathbb{R}\setminus\{-4\}$ $\quad f''(x) = \frac{(2x+8)(x+4)^{\cancel{2}} - (x^2+8x)2\cancel{(x+4)}}{(x+4)^{\cancel{4}3}} =$

$$= \frac{\cancel{2x^2}+\cancel{16x}+32-\cancel{2x^2}-\cancel{16x}}{(x+4)^3} = \frac{32}{(x+4)^3}$$



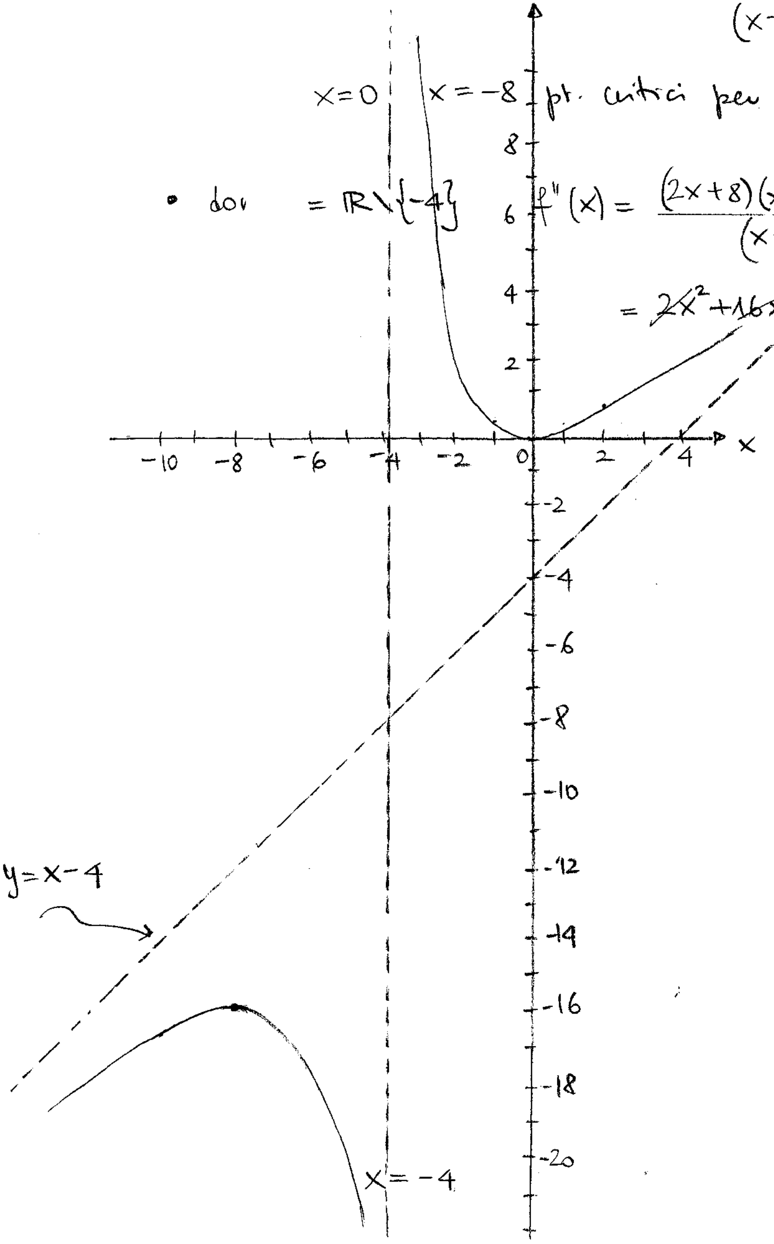


grafico qualitativo di f.

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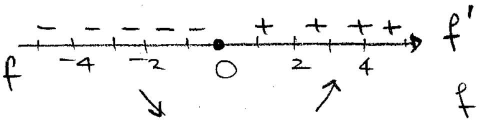
$$\boxed{f(x) = \frac{2x^2+1}{x^2+2}}$$

- $\mathrm{dom}\, f = \mathbb{R} =]-\infty, +\infty[$ $\qquad$ f è pari!
- segno di f : sempre positivo
- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 2$ $\Rightarrow$ $y=2$ asintoto orizzontale per f, per $x\to-\infty$ (per $x\to+\infty$).
- f è continua in tutto $\mathbb{R}$.

- dom $f' = \mathbb{R}$ $\quad f'(x) = \dfrac{4x(x^2+2)-(2x^2+1)2x}{(x^2+2)^2} = \dfrac{6x}{(x^2+2)^2}$

$f'(x)=0 \Leftrightarrow x=0$ pr. critico per f

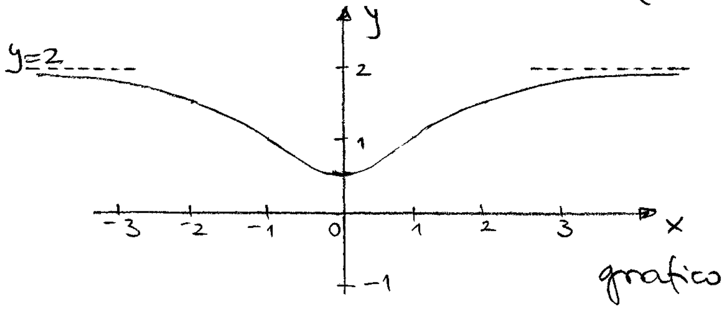


$f(0)=\frac{1}{2}$

$x=0$ pr. di min. loc. per f.

- dom $f'' = \mathbb{R}$ $\quad f''(x) = \dfrac{6(x^2+2)^{\cancel{2}}-6x\cdot 2(\cancel{x^2+2})2x}{(x^2+2)^{\cancel{4}\,3}} = \dfrac{6x^2+12-24x^2}{(x^2+2)^3}$

$= \dfrac{12-18x^2}{(x^2+2)^3} = \dfrac{6(2-3x^2)}{(x^2+2)^3} =$



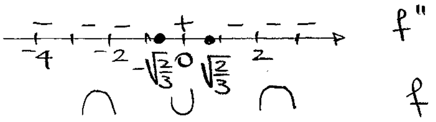
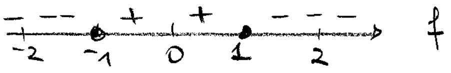


grafico qualitativo di f ☐

$$\boxed{f(x) = (1-x^2)^3}$$

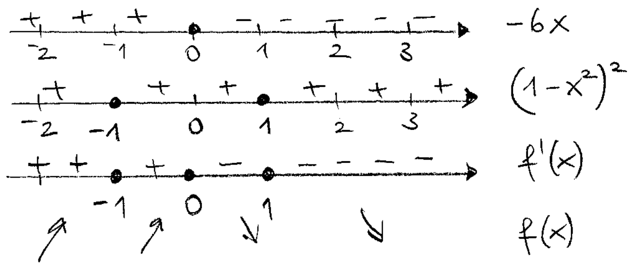
- dom $f = \mathbb{R} = \,]-\infty, +\infty[$ $\qquad f$ è pari!
- segno di f



- $\lim\limits_{x\to-\infty} f(x) = -\infty \qquad \lim\limits_{x\to+\infty} f(x) = -\infty$
- f è continua in $\mathbb{R}$ (essendo composizione di funzioni continue su $\mathbb{R}$)
- dom $f' = \mathbb{R}$ $\quad f'(x) = 3(1-x^2)^2\cdot(-2x)$

$= -6x(1-x^2)^2$

$f'(x)=0 \Leftrightarrow \begin{matrix} x=0 \\ x=\pm 1 \end{matrix}$ pr. critici per f



$x=0$ pr. di max. loc. per f $\qquad f(0)=1$

- dom $f'' = \mathbb{R}$ $\quad f''(x) = -6(1-x^2)^2 - 6x \cdot 2(1-x^2)(-2x)$

$$= -6(1-x^2)\left[(1-x^2) - 4x^2\right]$$
$$= 6(1-x^2)\left[5x^2-1\right]$$

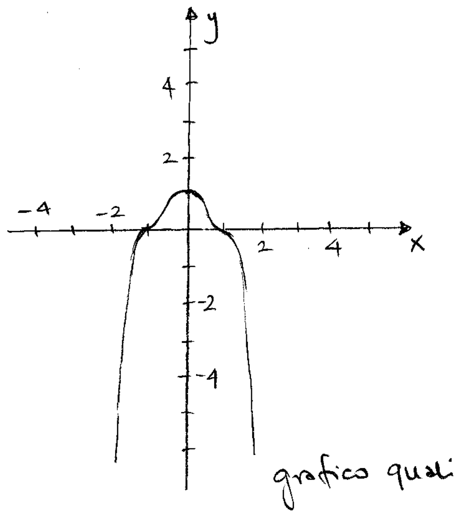
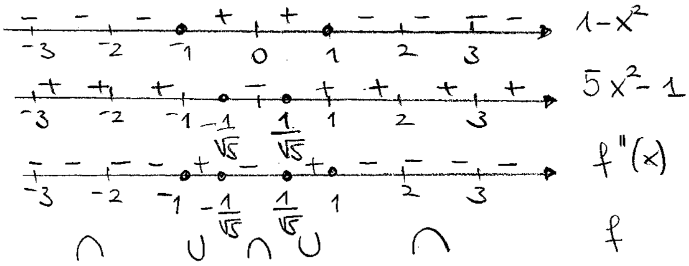


grafico qualitativo di f



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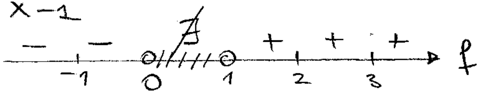
$$f(x) = \log\left(\frac{x}{x-1}\right)$$

- dom $f = \left\{x \in \mathbb{R} : \frac{x}{x-1} > 0\right\} = \underline{\underline{]-\infty, 0[\, \cup \,]1, +\infty[}}$

- segno di f : $f(x) = 0 \iff \frac{x}{x-1} = 1$ mai !

$$\frac{x}{x-1} > 1 \iff \frac{x}{x-1} - 1 > 0 \iff x > 1$$



- $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = \log 1 = 0$ $\quad y=0$ asintoto orizz. per f, per $x \to -\infty$ (per $x \to +\infty$)

  $\lim_{x \to 0^-} f(x) = -\infty$ $\quad x = 0$ asintoto verticale per f

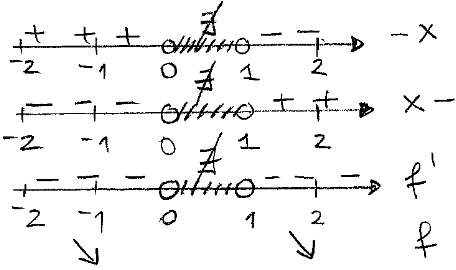
  $\lim_{x \to 1^+} f(x) = +\infty$ $\quad x = 1$ asintoto verticale per f

- f è continua nel suo dominio essendo rapporto e composizione di funzioni continue.

- dom $f' =]-\infty, 0[\, \cup \,]1, +\infty[$ $\quad f'(x) = \frac{\cancel{x-1}}{x} \cdot \frac{(x-1)-x}{(x-1)^2}$

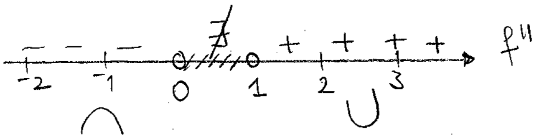
$$= \frac{-1}{x(x-1)}$$

f non ha pt. critici



- $\mathrm{dom}\, f'' = \,]-\infty, 0[\, \cup\,]1, +\infty[$

$$f''(x) = \frac{+[(x-1)+x]}{x^2(x-1)^2} = \frac{2x-1}{x^2(x-1)^2}$$



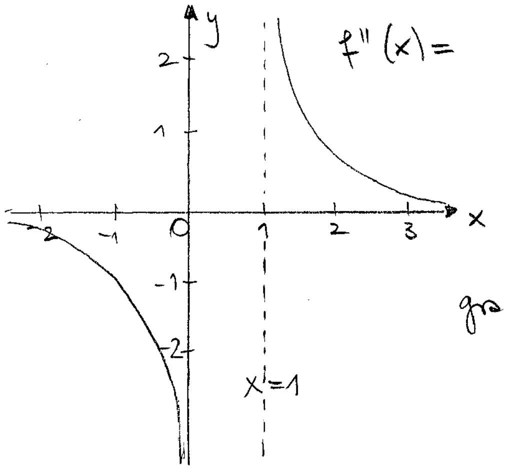


grafico qualitativo di f

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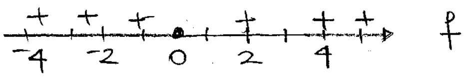
$$\boxed{f(x) = \frac{x^2}{e^x}} \qquad (= x^2 e^{-x})$$

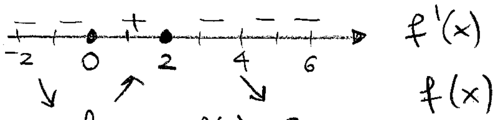
- $\mathrm{dom}\, f = \mathbb{R} = \,]-\infty, +\infty[$
- segno di f



- $\lim_{x \to -\infty} f(x) = +\infty \qquad \lim_{x \to +\infty} f(x) = 0$ y=0 asint. orizzontale per f per $x \to +\infty$.
- f è continua su ℝ essendo quoziente di funzioni continue su ℝ e il denominatore non si annulla mai.
- $\mathrm{dom}\, f' = \mathbb{R}$, $f'(x) = 2xe^{-x} - x^2e^{-x} = (-x^2+2x)e^{-x}$

$$= x(2-x)\, e^{-x}$$

$f'(x) = 0 \Leftrightarrow x=0,\ x=2$ pt. critici per f

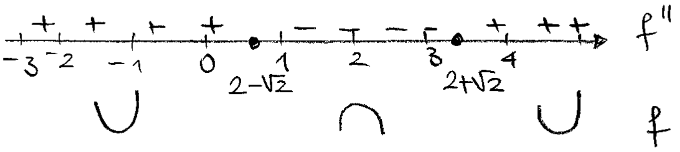


x=0 pt. di min. loc. per f $\qquad f(0)=0$

x=2 pt. di max. loc. per f $\qquad f(2) = 4e^{-2}$

- $\mathrm{dom}\, f'' = \mathbb{R}$, $f''(x) = (-2x+2)e^{-x} + (x^2-2x)e^{-x}$

$$= (x^2 - 4x + 2)\, e^{-x}$$



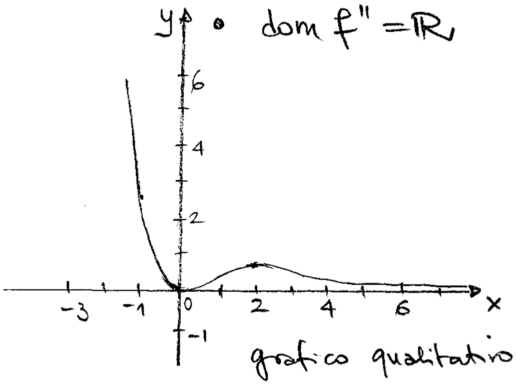


grafico qualitativo per f

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