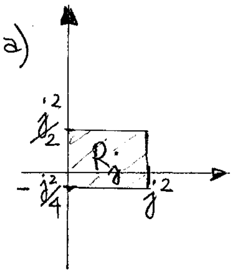
Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
Verifica settimanale di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2011 – 2012 — Rovereto, 5 – 9 dicembre 2011

1) i) $\frac{2}{2^x} - \frac{3}{4^x} + \frac{4}{8^x} - \dots + \frac{10}{512^x} = \sum_{n=1}^{9} (-1)^{n+1} \frac{(n+1)}{(2^n)^x}$;

$2 + \frac{4}{3} + \frac{6}{5} + \dots + \frac{30}{29} = \sum_{n=1}^{15} \frac{2n}{2n-1}$;

$\frac{1}{3}x - \frac{2}{6}x^2 + \frac{4}{9}x^3 - \dots - \frac{128}{24}x^8 = \sum_{n=1}^{8} \frac{(-2)^{n-1} x^n}{3n}$.

ii) a)



$\text{perimetro}(R_j) = 2j^2 + \cancel{2} \cdot \frac{3}{\cancel{4}_2} j^2 = \frac{7}{2} j^2$

$\text{area}(R_j) = j^2 \cdot \frac{3}{4} j^2 = \frac{3}{4} j^4$

$\sum_{j=1}^{5} \text{perimetro}(R_j) = \sum_{j=1}^{5} \frac{7}{2} j^2 = \frac{7}{2}(1+4+9+16+25) = \frac{7}{2} \cdot 55 = \underline{\underline{\frac{385}{2}}}$;

$\sum_{j=1}^{5} \text{area}(R_j) = \sum_{j=1}^{5} \frac{3}{4} j^4 = \frac{3}{4}(1+16+81+256+625) = 979 \cdot \frac{3}{4} = \underline{\underline{\frac{2937}{4}}}$.

b) $\sum_{n=3}^{8} f((-1)^n n) = f(-3) + f(4) + f(-5) + f(6) + f(-7) + f(8)$

$= 6 + 12 + 10 + 18 + 14 + 24 = \underline{\underline{84}}$.

c) $\sum_{m=1}^{5} (-1)^m \frac{m+1}{2m} = -\frac{\cancel{2}^1}{\cancel{2}} + \frac{3}{4} - \frac{\cancel{4}^2}{\cancel{6}_3} + \frac{5}{8} - \frac{\cancel{6}^3}{\cancel{10}_5} = -1 + \frac{3}{4} - \frac{2}{3} + \frac{5}{8} - \frac{3}{5} =$

$= \frac{-120 + 90 - 80 + 75 - 72}{120} = -\frac{107}{120}$;

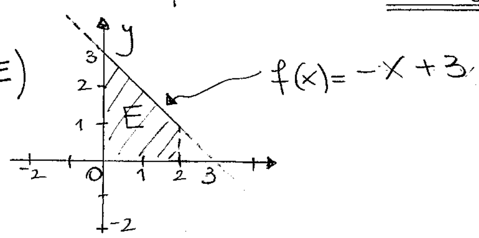
$\sum_{k=2}^{39} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2} \right) = \left(\frac{1}{2^2} - \cancel{\frac{1}{3^2}} \right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}} \right) + \dots + \left(\cancel{\frac{1}{38^2}} - \cancel{\frac{1}{39^2}} \right) + \left(\cancel{\frac{1}{39^2}} - \frac{1}{40^2} \right) =$

$= \frac{1}{4} - \frac{1}{40^2} = \frac{1}{4} - \frac{1}{1600} = \underline{\underline{\frac{399}{1600}}}$

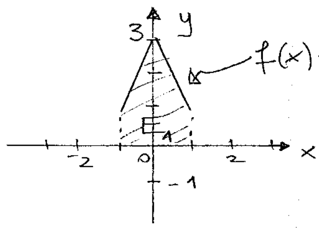
2) $\int_0^2 (-x+3)\,dx = \text{area}(E)$

$= \frac{(3+1)}{\cancel{2}} \cdot \cancel{2}$

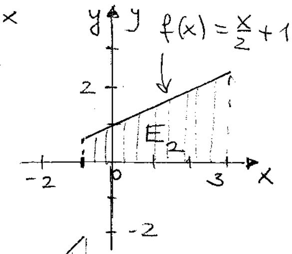
$= \underline{\underline{4}}$



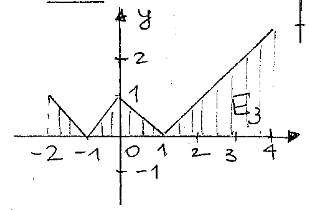
$$\int_{-1}^{1}(-2|x|+3)\,dx =$$



$$\text{area}(E_1) = \cancel{2}\left(\frac{3+1}{\cancel{2}}\right)\cdot 1 = \underline{\underline{4}}$$

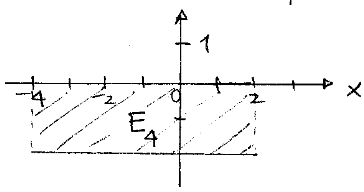


$$\int_{-1}^{3}\left(\frac{x}{2}+1\right)dx = \text{area}(E_2) = \left(\frac{1}{2}+\frac{5}{2}\right)\cdot 2 = \underline{\underline{6}}.$$



$$\int_{-2}^{4}\big|1-|x|\big|\,dx = \text{area}(E_3) = \frac{1}{2}+1+\frac{9}{2} = \underline{\underline{6}}.$$

$$\int_{-4}^{2}(-2)\,dx = -\,\text{area } E_4 = -6\cdot 2 = \underline{\underline{-12}}$$



3) i) $\displaystyle\int_{-1}^{0}(2^x-1)\,dx = \left[\frac{2^x}{\log 2}-x\right]_{-1}^{0} = \left(\frac{1}{\log 2}-0\right)-\left(\frac{2^{-1}}{\log 2}+1\right) = \underline{\underline{\frac{1}{2\log 2}-1}};$

$$\int_{1}^{2}\left(x^{-3}+\frac{1}{x}\right)dx = \left[\frac{x^{-2}}{-2}+\log|x|\right]_{1}^{2} = \left(-\frac{1}{8}+\log 2\right)-\left(-\frac{1}{2}\right) = \underline{\underline{\frac{3}{8}+\log 2}};$$

$$\int_{0}^{1}\frac{x^2-1}{x+1}\,dx = \int_{0}^{1}(x-1)\,dx = \left[\frac{x^2}{2}-x\right]_{0}^{1} = \frac{1}{2}-1 = \underline{\underline{-\frac{1}{2}}};$$

ii) $\displaystyle\int_{1}^{2}\left(\sqrt[3]{x}+3\sqrt{x}\right)dx = \left[\frac{3}{4}x^{\frac{4}{3}}+\cancel{3}\,x^{\frac{3}{2}}\cdot\frac{2}{\cancel{3}}\right]_{1}^{2} = \left(\frac{3}{4}\cdot 2^{\frac{4}{3}}+2\cdot 2^{\frac{3}{2}}\right)-\left(\frac{3}{4}+2\right)$

$$= \underline{\underline{\frac{3}{2}\sqrt[3]{2}+4\sqrt{2}-\frac{11}{4}}};$$

$$\int_{1}^{2}\left(\frac{x^3-x^4}{x^2}\right)dx = \int_{1}^{2}(x-x^2)\,dx = \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{1}^{2} = \left(2-\frac{8}{3}\right)-\left(\frac{1}{2}-\frac{1}{3}\right) =$$

$$= \frac{3}{2}-\frac{7}{3} = \frac{9-14}{6} = \underline{\underline{-\frac{5}{6}}};$$

$$\int_{-2}^{-1}\left(e^x+\frac{1}{x}\right)dx = \left[e^x+\log|x|\right]_{-2}^{-1} = \left(e^{-1}+\cancel{\log 1}\right)-\left(e^{-2}+\log 2\right) =$$

$$= \underline{\underline{\frac{1}{e}-\frac{1}{e^2}-\log 2}}.$$

4) a) $F(x) = (x^2-x+1)e^x+8$ è una funzione derivabile su tutto $\mathbb{R}$.

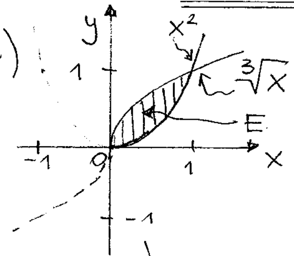
Inoltre $F'(x) = (2x-1)e^x+(x^2-x+1)e^x = e^x(x^2+x) = f(x)\quad \forall x\in\mathbb{R}$.

Quindi F è una primitiva di f.

$$\int_{-4}^{2} f(x)\,dx = \Big[F(x)\Big]_{-4}^{2} = \Big[(x^2-x+1)e^{x}+8\Big]_{-4}^{2} =$$
$$= \Big[(4-2+1)e^{2}+8\Big] - \Big[(16+4+1)e^{-4}+8\Big] =$$
$$= \underline{\underline{3e^{2}-21e^{-4}}}\,.$$

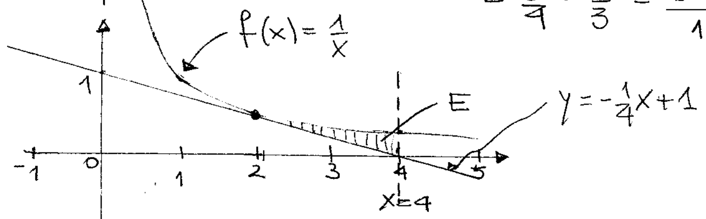
$\square$

b) i)



$$\text{area}(E) = \int_{0}^{1}\left(\sqrt[3]{x} - x^2\right)dx = \left[\frac{3}{4}\cdot x^{\frac{4}{3}} - \frac{x^3}{3}\right]_{0}^{1} =$$
$$= \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \underline{\underline{\frac{5}{12}}}\ ;$$

ii)



$$f(x) = \frac{1}{x} \qquad f'(x) = -\frac{1}{x^2} \Rightarrow f'(2) = -\frac{1}{4} \qquad y = \frac{1}{2} - \frac{1}{4}(x-2)$$
$$\underline{\underline{y = -\frac{1}{4}x + 1}}\,.$$

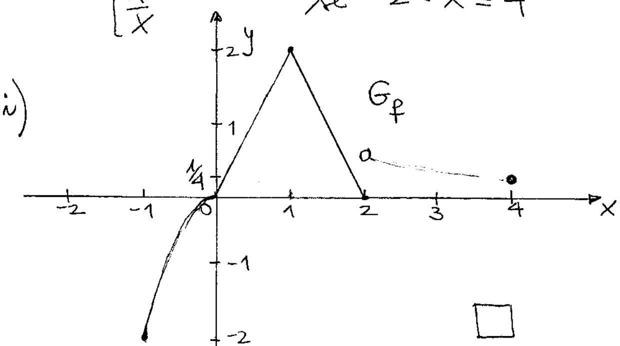
$$\text{area}(E) = \int_{2}^{4}\left(\frac{1}{x} - \left(-\frac{1}{4}x+1\right)\right)dx =$$
$$= \int_{2}^{4}\left(\frac{1}{x} + \frac{1}{4}x - 1\right)dx = \left[\log|x| + \frac{1}{4}\cdot\frac{x^2}{2} - x\right]_{2}^{4} = (\log 4 + 2 - 4) -$$
$$-\left(\log 2 + \frac{1}{2} - 2\right) = \log 4 - \log 2 - 2 + \frac{3}{2} = \underline{\underline{\log 2 - \frac{1}{2}}}$$

$\blacksquare$

5) $f: [-1,4] \to \mathbb{R}$

$$f(x) = \begin{cases} 2x^3 & \text{se } -1 \le x \le 0 \\ -2|x-1|+2 & \text{se } 0 < x \le 2 \\ \frac{1}{x} & \text{se } 2 < x \le 4 \end{cases}$$

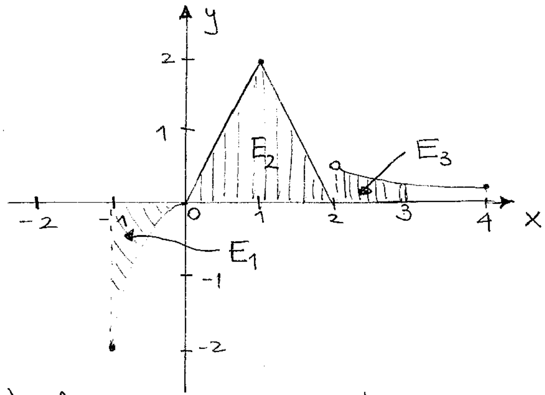
i)



$\square$

ii)
$$\int_{-1}^{3} f(x)\,dx = 2\int_{-1}^{0} x^3\,dx + \int_{0}^{2}\left[-2|x-1|+2\right]dx + \int_{2}^{3}\frac{1}{x}\,dx$$
$$= 2\left[\frac{x^4}{4}\right]_{-1}^{0} + \frac{2\cdot\cancel{2}}{\cancel{2}} + \Big[\log|x|\Big]_{2}^{3} =$$
$$= -\cancel{2}\cdot\frac{1}{\cancel{4}_2} + 2 + \log 3 - \log 2 =$$
$$= \underline{\underline{\frac{3}{2} + \log\frac{3}{2}}}\,.$$

$\square$



$$\int_{-1}^{3} f(x)\,dx = -\,\text{area}(E_1) + \text{area}(E_2) + \text{area}(E_3).$$

$\square$

iv) f non è continua in $x = 2$ e quindi f non soddisfa le ipotesi del teorema di Weierstrass su $[-1,3]$.

$x = -1$, $x = 2$, $x = 4$ sono pt. di minimo locali per f.

$x = 1$ è un punto di max. loc. per f.

Osserviamo che $x = -1$ è pt. di minimo di f su $[-1,4]$;

$x = 1$ è pt. di massimo di f su $[-1,4]$. $\blacksquare$

6) $\boxed{f(x) = \dfrac{1}{e^x x^2}}$

• $\text{dom} f = \mathbb{R}\setminus\{0\} =]-\infty,0[\,\cup\,]0,+\infty[$

• segno di f : $f > 0$ su $\mathbb{R}\setminus\{0\}$

• $\lim_{x\to-\infty} f(x) = +\infty$ $\quad \lim_{x\to+\infty} f(x) = 0 \Rightarrow y = 0$ è asintoto orizzontale per f, per $x \to +\infty$.

$\lim_{x\to0^-} f(x) = +\infty$ $\quad \lim_{x\to0^+} f(x) = +\infty \Rightarrow x = 0$ è asintoto verticale per f.

• f è continua su $\mathbb{R}\setminus\{0\}$.

• $\text{dom} f' = \mathbb{R}\setminus\{0\}$ $\quad f'(x) = \dfrac{-\cancel{e^x}x^2 - \cancel{e^x}\cdot 2x}{e^{\cancel{2}x}x^4} = \dfrac{-(x^2+2x)}{e^x\cdot x^4} = \dfrac{-(x+2)}{e^x x^3}$

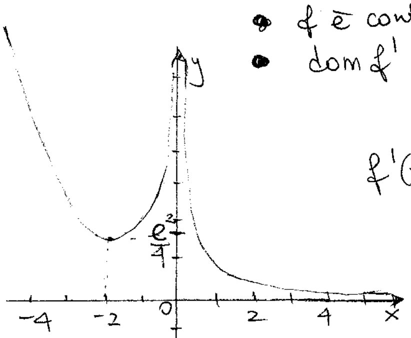
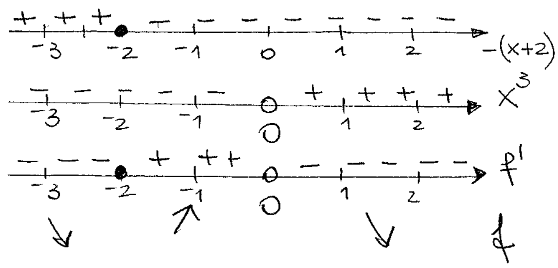


grafico qualitativo di f

$f'(x) = 0 \Leftrightarrow x = -2$ pt. critico per f $\quad f(-2) = \dfrac{e^2}{4}$

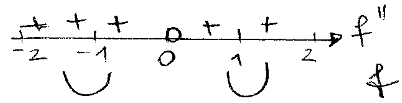


$x = -2$ è pt. di min. loc. stretto per f

• $\text{dom} f'' = \mathbb{R}\setminus\{0\}$

$$f''(x) = \frac{-1\cdot\cancel{e^x}x^{\cancel{3}} + (x+2)\left[\cancel{e^x}x^{\cancel{3}} + \cancel{e^x}\cdot 3\cancel{x^2}\right]}{e^{\cancel{2}x}\cdot x^{\cancel{6}4}} = \frac{-x + (x+2)(x+3)}{e^x\cdot x^4}$$

$$= \frac{x^2+4x+6}{e^x\cdot x^4}$$



$\blacksquare$

$\boxed{f(x) = \dfrac{1}{x^2 - x}}$

- $\mathrm{dom}\, f = \mathbb{R}\setminus\{0,1\} =]-\infty, 0[\, \cup\,]0,1[\, \cup\,]1, +\infty[$
- segno di f

$$\overset{+\;+\;+\;0\;-\;0\;+\;+}{\overline{\quad -1 \quad 0 \quad 1 \quad 2 \quad}}\; x^2 - x \quad (\text{segno di } f)$$

- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0 \qquad y = 0$ a. orizz. per f per $x \to -\infty$ e per $x \to +\infty$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x(x-1)} = +\infty \qquad \lim_{x\to 0^+} \frac{1}{x(x-1)} = -\infty \qquad x = 0 \text{ a. verticale}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1}{x(x-1)} = -\infty \qquad \lim_{x\to 1^+} \frac{1}{x(x-1)} = +\infty \qquad x = 1 \text{ a. verticale}$$

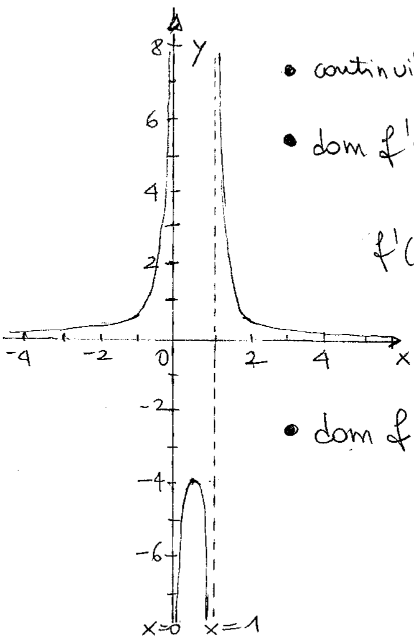


grafico qualitativo di f

- continuità di f in $\mathbb{R}\setminus\{0,1\}$
- $\mathrm{dom}\, f' = \mathbb{R}\setminus\{0,1\} \qquad f'(x) = \dfrac{-(2x-1)}{(x^2-x)^2}$

$f'(x) = 0 \iff x = \frac{1}{2}$ pt. di max loc. stretto

$$f\left(\tfrac{1}{2}\right) = \frac{1}{\frac{1}{4} - \frac{1}{2}} = -4$$

$$\overset{+\;+\;+\;+\;\bullet\;-\;-\;-\;-}{\overline{\quad -1 \quad 0 \quad 1 \quad 2 \quad}}\; -(2x-1)$$

$$\overset{+\;+\;0\;+\;0\;+\;+}{\overline{\quad -1 \quad 0 \quad 1 \quad 2 \quad}}\; (x^2-x)^2$$

$$\overset{+\;+\;+\;0\;+\;\bullet\;-\;0\;-\;-\;-}{\overline{\quad -1 \quad 0 \quad 1 \quad 2 \quad}}\; f'(x)$$

$$\nearrow \quad \nearrow \searrow \quad \searrow \qquad f$$

- $\mathrm{dom}\, f'' = \mathbb{R}\setminus\{0,1\}$

$$f''(x) = \frac{-2(x^2-x)^{\cancel{2}} + (2x-1)^2\, 2\,\cancel{(x^2-x)}}{(x^2-x)^{\cancel{4}\,3}} = \frac{-2x^2 + 2x + 8x^2 - 8x + 2}{(x^2-x)^3} =$$

$$= \frac{6x^2 - 6x + 2}{(x^2-x)^3} \qquad \text{quindi } f'' \text{ ha lo stesso segno di } f$$

$$\overset{+\;+\;+\;0\;-\;0\;+\;+}{\overline{\quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad}}\; f''$$

$$\cup \qquad \cap \qquad \cup \qquad f$$

$\square$

$\boxed{f(x) = \log\left(1 + \dfrac{1}{x^2}\right)}$

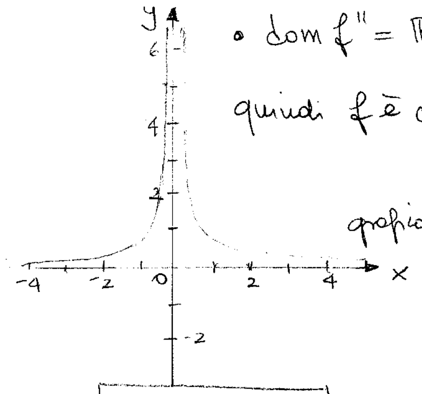
f è pari !!

- $\mathrm{dom}\, f = \mathbb{R}\setminus\{0\} =]-\infty, 0[\, \cup\,]0, +\infty[$
- segno di $f \quad > 0$ poiché $1 + \frac{1}{x^2} > 1$ e $\log x > 0$ se $x > 1$.
- $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0 \qquad y = 0$ asint. orizz. per f per $x \to -\infty$ per $x \to +\infty$
- $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = +\infty$
- continuità di f in $\mathbb{R}\setminus\{0\}$.
- $\mathrm{dom}\, f' = \mathbb{R}\setminus\{0\} \qquad f'(x) = \dfrac{1}{1 + \frac{1}{x^2}} \cdot \dfrac{-2}{x^3} = \dfrac{-2}{x(x^2+1)}$

$$\overset{+\;+\;+\;+\;0\;-\;-\;-\;-\;-}{\overline{\quad -4 \quad -2 \quad 0 \quad 2 \quad 4 \quad}}\; f'(x)$$

$$\nearrow \qquad \searrow \qquad f(x)$$

non ci sono pt. critici per f e quindi non ci sono max o min. loc. di f



- dom $f'' = \mathbb{R}\setminus\{0\}$ $\qquad f''(x) = \dfrac{2\left[(x^2+1)+x\cdot 2x\right]}{x^2(1+x^2)^2} = \dfrac{2(3x^2+1)}{x^2(1+x^2)^2} > 0 \quad \forall x \neq 0$

quindi f è convessa su $]-\infty, 0[$ ed è convessa su $]0, +\infty[$.

grafico qualitativo per f

$\square$

$$\boxed{f(x) = (x^2+x)^3}$$

- dom $f = \mathbb{R} =]-\infty, +\infty[$
- segno di f $\qquad$ + + • − • + + + $\quad f(x)$ $\qquad$ (-2, -1, 0, 1, 2)
- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = +\infty$
- continuità di f su $\mathbb{R}$

- dom $f' = \mathbb{R}$ $\qquad f'(x) = 3(x^2+x)^2(2x+1)$ $\qquad$ − • − • + • + + + $\quad f'$ $\qquad$ (-2, -1, $-\frac{1}{2}$, 0, 1, 2, x) $\quad$ ↘ ↗ $\quad f$

$f'(x) = 0 \qquad x = -\frac{1}{2}$ pt. critico per f

Segue che $x = -\frac{1}{2}$ è pt. di min. loc. per f e

$$f\left(-\tfrac{1}{2}\right) = \left(\tfrac{1}{4} - \tfrac{1}{2}\right)^3 = -\tfrac{1}{64}$$

Inoltre, $x=-1$, $x=0$ sono pt. critici per f, ma non sono né pt. di minimo loc. né pt. di massimo loc. per f.

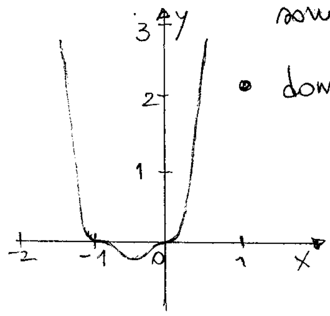


grafico qualitativo di f

- dom $f'' = \mathbb{R}$ $\qquad f''(x) = 6(x^2+x)(2x+1)^2 + 3(x^2+x)^2\cdot 2$

$$= (x^2+x)\left[24x^2+24x+6+6x^2+6x\right]$$

$$= (x^2+x)\left[30x^2+30x+6\right] = 6(x^2+x)(5x^2+5x+1)$$

$$5x^2+5x+1=0 \iff x_{1/2} = \frac{-5\pm\sqrt{25-20}}{10} = \frac{-5\pm\sqrt{5}}{10}$$

+ + • − • + + $\quad x^2+x$ $\qquad$ (-2, -1, 0, 1) $\qquad 2=\sqrt{4} < \sqrt{5} < \sqrt{9} = 3$

+ + + + • − • + + + + $\quad 5x^2+5x+1$ $\qquad -\frac{3}{10} < \frac{-5+\sqrt{5}}{10} < -\frac{2}{10}$

+ • − • + • − • + + + $\quad f''$ $\qquad -\frac{8}{10} < \frac{-5-\sqrt{5}}{10} < -\frac{7}{10}$

$\cup \quad \cap \cup \cap \quad \cup$ $\quad f$