Università degli studi di Trento - Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione - CdL in Filosofia
SECONDA PROVA INTERMEDIA DI ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2011-2012 - Rovereto, 9 gennaio 2012

FILA (A)

1) $$\frac{x^2-5|x|}{|x^2-1|} \leq 0 \iff \begin{cases} x^2-5|x| \leq 0 \\ x^2 \neq 1 \end{cases} \iff \begin{cases} x \geq 0 \\ x^2-5x \leq 0 \\ x \neq 1 \end{cases} \text{ o } \begin{cases} x<0 \\ x^2+5x \leq 0 \\ x \neq -1 \end{cases}$$

$$\iff \underline{\underline{S = [-5,5] \setminus \{-1,1\}}}.$$ ☐

$$2^{x^2-3x}\left(\log_3 \frac{1}{9} + \log_{\sqrt{2}} 64\right) \geq 1 \iff 2^{x^2-3x}(-2+6) \geq 1 \iff 2^{x^2-3x} \cdot 4 \geq 1$$

$$\iff 2^{x^2-3x+2} \geq 2^0 \underset{2^x \uparrow}{\iff} x^2-3x+2 \geq 0$$

$$\underline{\underline{S =]-\infty, 1] \cup [2, +\infty[}}.$$ ☐

$$\log_2(|x-2|-1) < 0 \iff 0 < |x-2|-1 < 1 \iff 1 < |x-2| < 2$$

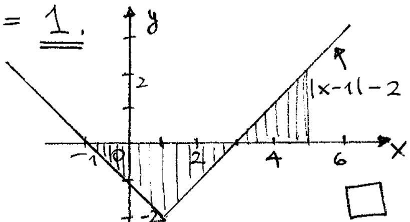
$$\iff \begin{cases} x-2<-1 \text{ o } x-2>1 \\ -2 < x-2 < 2 \end{cases} \iff \begin{cases} x<1 \text{ o } x>3 \\ 0<x<4 \end{cases}$$

$$\underline{\underline{S =]0,1[\cup]3,4[}}.$$ ■

2) i) $$\lim_{x \to 0^-} \frac{1}{|x|-x^2} = \lim_{x \to 0^-} \frac{1}{-x-x^2} = \lim_{x \to 0^-} \frac{1}{\underbrace{-x}_{\text{infinitesimo positivo}}\underbrace{(1+x)}_{\to 1}} = \underline{\underline{+\infty}}.$$

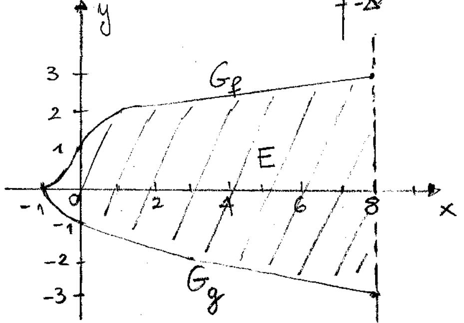
$$\int_1^e 3\frac{(\log x)^2}{x}\,dx = \cancel{3}\left[\frac{(\log x)^3}{\cancel{3}}\right]_1^e = (\log e)^3 - (\log 1)^3 = \underline{\underline{1}}.$$

$$\int_{-1}^{5} (|x-1|-2)\,dx = -\frac{4 \cdot \cancel{2}}{\cancel{2}} + \frac{2 \cdot \cancel{2}}{\cancel{2}} = \underline{\underline{-2}}.$$ ☐



ii) $f, g : [-1, 8] \to \mathbb{R}$

$f(x) = \sqrt[3]{x} + 1$, $g(x) = -\sqrt{x+1}$



$$\text{area}(E) = \int_{-1}^{8} (f(x)-g(x))\,dx =$$

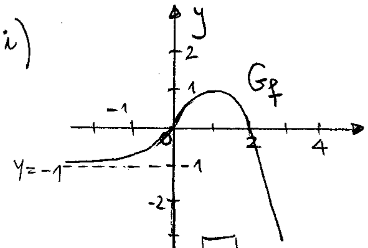
$$\int_{-1}^{8}\left(\sqrt[3]{x}+1+\sqrt{x+1}\right)dx = \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + x + \frac{(x+1)^{3/2}}{3/2}\right]_{-1}^{8} = \left[\frac{3}{4}\,8^{4/3} + 8 + \frac{2}{3} \cdot 9^{3/2}\right] -$$

$$- \left[\frac{3}{4}(-1)^{4/3} - 1 + \frac{2}{3} \cdot 0\right] = [12+8+18] - \left[\frac{3}{4}-1\right] = 39 - \frac{3}{4} = \frac{153}{4}.$$ ☐

iii) $$-\frac{2}{x^2+1} + \frac{4}{x^5+1} - \frac{8}{x^8+1} + \cdots + \frac{256}{x^{23}+1} = \underline{\underline{\sum_{n=1}^{8} \frac{2^n \cdot (-1)^n}{x^{3n-1}+1}}}.$$ ■

3) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} e^x - 1 & \text{se } x < 0 \\ -x^2 + 2x & \text{se } x \geq 0 . \end{cases}$

i)



$\square$

$$-x^2+2x = -(x^2-2x) = -(x-1)^2+1$$

ii) $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x \to 0^-} (e^x - 1) = 0$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x \to 0^+} (-x^2+2x) = 0 = f(0)$

$\Rightarrow \lim\limits_{x\to 0} f(x) = 0 = f(0)$. Quindi f è continua in $x=0$. $\square$

iii) $\lim\limits_{h\to 0^-} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0^-} \dfrac{(e^h-1)-0}{h} = 1$ ↖ limite notevole

$\lim\limits_{h\to 0^+} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0^+} \dfrac{(-h^2+2h)-0}{h} = 2$

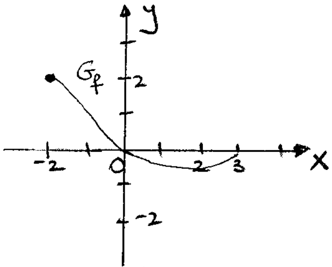
Allora $\nexists \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h}$ e quindi f non è derivabile in $x=0$. $\blacksquare$

4) $f : [-2,3] \to \mathbb{R}$ continua t.c.

i) $f(-2) = 2$, f non è decrescente su $[-2,3]$;

ii) $F(x) = \int_{-2}^{x} f(t)\,dt$ è positiva su $]-2,3]$;

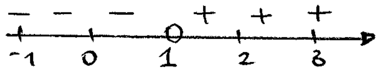
F è crescente su $[-2,0]$ e decrescente su $[0,3]$



questo è un esempio di funzione f soddisfacente i) ii) $\blacksquare$

5) $f(x) = \dfrac{e^x}{(x-1)^3}$

- $\operatorname{dom} f = \mathbb{R} \setminus \{1\} =]-\infty, 1[\cup]1, +\infty[$
- segno di f :



Nota: $e^x > 0$ $\forall x \in \mathbb{R}$

- $\lim\limits_{x\to -\infty} f(x) = 0$ $\qquad \lim\limits_{x\to 1^-} f(x) = -\infty$ $\qquad x=1$ asintoto verticale per f

$y=0$ asintoto orizzontale per f, per $x \to -\infty$ $\qquad \lim\limits_{x\to 1^+} f(x) = +\infty$

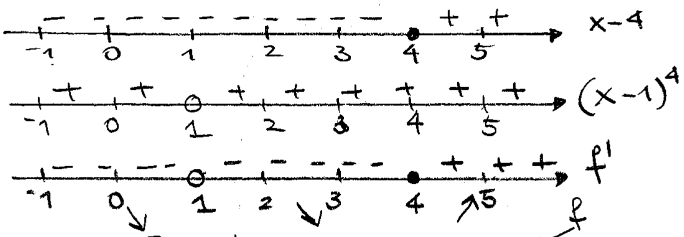
$\lim\limits_{x\to +\infty} f(x) = +\infty$.

- f è continua in $\mathbb{R}\setminus\{1\}$ essendo composizione e rapporto di funzioni continue.

- dom $f' = \mathbb{R}\setminus\{1\}$ $\quad f'(x) = \dfrac{e^x(x-1)^{\cancel{3}} - e^x \cdot 3\cancel{(x-1)^2}}{(x-1)^{\cancel{6}4}} = \dfrac{e^x(x-4)}{(x-1)^4}$

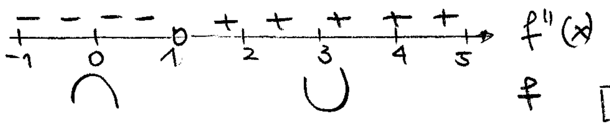
$x=4$ pt. critico per f e $\quad f(4) = \dfrac{e^4}{3^3} = \dfrac{e^4}{27} \sim 2$

$x=4$ è pt. di minimo locale stretto per f



- dom $f'' = \mathbb{R}\setminus\{1\}$ $\quad f''(x) = \dfrac{[e^x(x-4)+e^x](x-1)^{\cancel{4}} - e^x(x-4)4\cancel{(x-1)^3}}{(x-1)^{\cancel{8}5}}$

$$= \frac{e^x[(x-3)(x-1)-4(x-4)]}{(x-1)^5} = \frac{\overset{>0}{e^x}\overset{>0}{(x^2-8x+19)}}{(x-1)^5}$$



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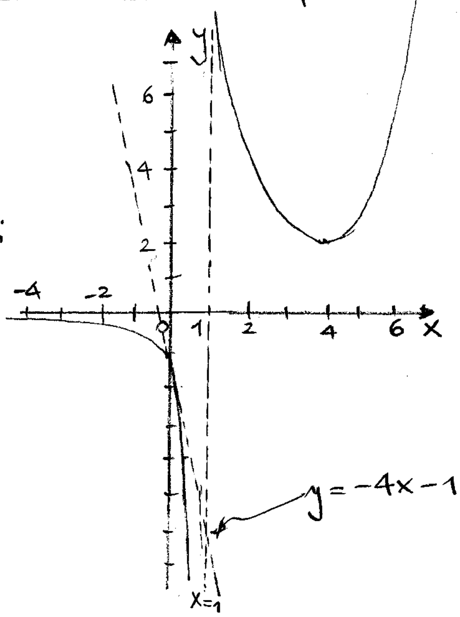
ii) $y = -1 - 4(x-0) \Rightarrow \underline{y=-4x-1}$ è l'eq. della retta tangente al grafico di f nel pt. $(0,-1)$.

grafico qualitativo di f



iii) $\displaystyle\int_3^4 \frac{e^x}{f(x)}\,dx = \int_3^4 (x-1)^3\,dx$

$$= \left[\frac{(x-1)^4}{4}\right]_3^4 = \frac{3^4}{4} - \frac{2^4}{4}$$

$$= \frac{81}{4} - \frac{16}{4} = \underline{\underline{\frac{65}{4}}}.$$

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■

6) $C_{n,2} = 21 \iff \dfrac{n(n-1)\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}} = 21 \iff n^2 - n - 42 = 0$

$\iff n_{1/2} = \dfrac{1 \pm \sqrt{1+168}}{2} = \dfrac{1 \pm \sqrt{169}}{2} = \dfrac{1 \pm 13}{2} \longrightarrow \underline{\underline{n=7}}.$

■

FILA Ⓑ

1) $\frac{x^2-6|x|}{|x^2-2|} \leq 0 \iff \begin{cases} x^2-6|x|\leq 0 \\ x^2\neq 2 \end{cases} \iff \begin{cases} x\geq 0 \\ x^2-6x\leq 0 \\ x\neq\sqrt{2} \end{cases}$ o $\begin{cases} x<0 \\ x^2+6x\leq 0 \\ x=-\sqrt{2} \end{cases}$

$\Rightarrow \underline{\underline{S=[-6,6]\setminus\{-\sqrt{2},\sqrt{2}\}.}}$ ☐

$3^{x^2+3x}\left(\log_3 27+\log_{\frac{1}{2}}\frac{1}{64}\right)\leq 1 \iff 3^{x^2+3x}(3+6)\leq 1 \iff 3^{x^2+3x}\cdot 9\leq 1$

$\iff 3^{x^2+3x+2}\leq 3^0 \underset{3^x\nearrow}{\iff} x^2+3x+2\leq 0$

$\Rightarrow \underline{\underline{S=[-2,-1].}}$ ☐

$\log_3(|x-3|-2)<0 \iff 0<|x-3|-2<1 \iff 2<|x-3|<3$

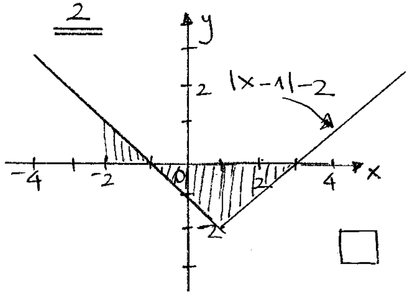
$\iff \begin{cases} x-3<-2 \text{ o } x-3>2 \\ -3<x-3<3 \end{cases} \iff \begin{cases} x<1 \text{ o } x>5 \\ 0<x<6 \end{cases}$

$\Rightarrow \underline{\underline{S=]0,1[\,\cup\,]5,6[\,.}}$ ■

2) i) $\lim_{x\to 0^-}\frac{1}{\boxed{x^2+|x|}} = +\infty.$ infinitesimo positivo

$\int_1^e 2\,\frac{(\log x)^3}{x}\,dx = \cancel{2}\left[\frac{(\log x)^4}{\cancel{4}\,2}\right]_1^e = \frac{(\log e)^4}{2}-\frac{(\log 1)^4}{2} = \underline{\underline{\frac{1}{2}}}.$

$\int_{-2}^{3}(|x-1|-2)\,dx = \frac{1}{2}-\frac{4\cdot\cancel{2}}{\cancel{2}} = \underline{\underline{-\frac{7}{2}}}.$



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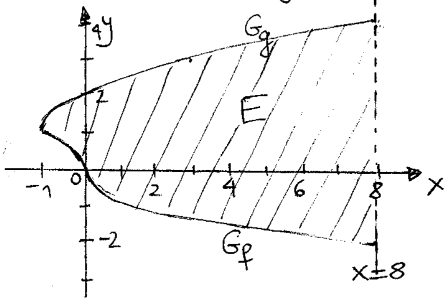
ii) $f,g:[-1,8]\to\mathbb{R}$

$f(x)=-\sqrt[3]{x} \qquad g(x)=\sqrt{x+1}+1$



$\text{area}(E)=\int_{-1}^{8}(g(x)-f(x))\,dx=\int_{-1}^{8}\left(\sqrt{x+1}+1+\sqrt[3]{x}\right)dx=$

$=\left[\frac{(x+1)^{3/2}}{3/2}+x+\frac{x^{4/3}}{4/3}\right]_{-1}^{8}=\left[\frac{2}{3}\cdot 9^{3/2}+8+\frac{3}{4}\cdot 8^{4/3}\right]-$

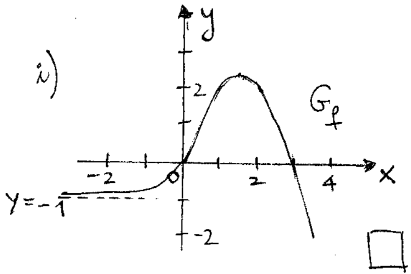
$-\left[\frac{2}{3}\cdot 0-1+\frac{3}{4}(-1)^{4/3}\right]=\left[\frac{2}{\cancel{3}}\cdot\overset{9}{\cancel{27}}+8+\frac{3}{\cancel{4}}\cdot\overset{4}{\cancel{16}}\right]$

$-\left[-1+\frac{3}{4}\right]=[18+8+12]+1-\frac{3}{4}=39-\frac{3}{4}=\frac{153}{4}$ ☐

iii) $\frac{2}{x^4+3}-\frac{4}{x^7+3}+\frac{8}{x^{10}+3}-\cdots-\frac{256}{x^{25}+3}=\sum_{n=1}^{8}\frac{2^n(-1)^{n+1}}{x^{3n+1}+3}.$ ■

3) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} e^x - 1 & \text{se } x<0 \\ -x^2+3x & \text{se } x \geq 0 . \end{cases}$

i)



□

$$-x^2+3x = -(x^2-3x) = -(x-\tfrac{3}{2})^2+\tfrac{9}{4}$$

ii) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (e^x-1) = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-x^2+3x) = 0 = f(0)$

$\Rightarrow \lim_{x\to 0} f(x) = 0 = f(0)$. Quindi f è continua in $x=0$.

□

iii) $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(e^h-1)-0}{h} = 1$ ↳ limite notevole

$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(-h^2+3h)-0}{h} = 3$.

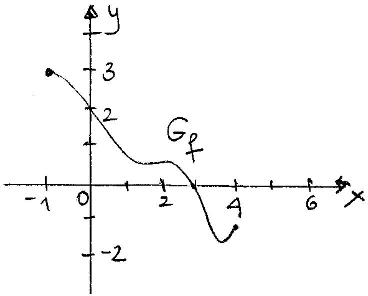
Quindi $\nexists \lim_{h\to 0} \frac{f(h)-f(0)}{h}$ e quindi f non è derivabile in $x=0$. ■

4) $f: [-1,4] \to \mathbb{R}$ continua t.c.

i) $f(-1)=3$, f non è decrescente su $[-1,4]$;

ii) $F(x) = \int_{-1}^{x} f(t)\,dt$ è positiva su $]-1,4]$;

$F(x)$ è crescente su $[-1,3]$ e decrescente su $[3,4]$.

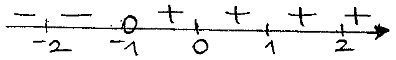


questo è un esempio di funzione f soddisfacente i); ii)

■

5) $f(x) = \dfrac{e^x}{(x+1)^3}$

- $\text{dom} f = \mathbb{R}\setminus\{-1\} =]-\infty,-1[\cup]-1,+\infty[$
- segno di f :



Nota: $e^x > 0 \ \forall x \in \mathbb{R}$

- $\lim_{x\to -\infty} f(x) = 0$ $\quad y=0$ asintoto orizzontale per f, per $x \to -\infty$

$\lim_{x\to -1^-} f(x) = -\infty$ $\quad x=-1$ asintoto verticale per f

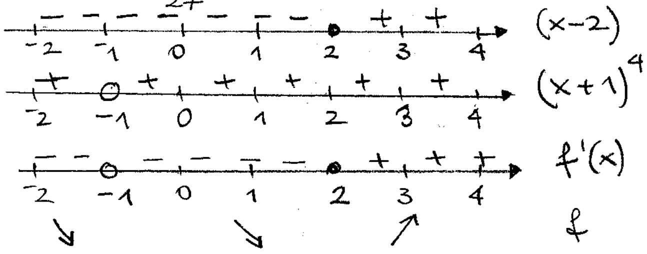
$\lim_{x\to -1^+} f(x) = +\infty$

$\lim_{x\to +\infty} f(x) = +\infty$.

- f è continua in $\mathbb{R}\setminus\{-1\}$ essendo composizione e rapporto di funzioni continue.

- dom $f' = \mathbb{R}\setminus\{-1\}$ $\quad f'(x) = \dfrac{e^x(x+1)^{\cancel{3}} - e^x\cdot 3\cancel{(x+1)}^2}{(x+1)^{\cancel{6}4}} = \dfrac{e^x(x-2)}{(x+1)^4}$

$x=2$ pt. critico per f e $f(2) = \dfrac{e^2}{27} \sim 0.2$

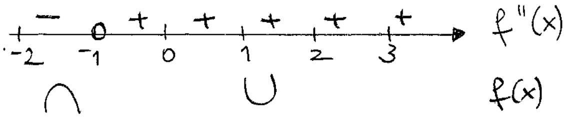


$x=2$ ē pt. di minimo locale stretto per f

- dom $f'' = \mathbb{R}\setminus\{-1\}$

$$f''(x) = \frac{\left(e^x(x-2)+e^x\right)(x+1)^{\cancel{4}} - e^x(x-2)\cdot 4\cancel{(x+1)}^3}{(x+1)^{\cancel{8}5}}$$

$$= \frac{e^x\left[x^2-1-4x+8\right]}{(x+1)^5} = \frac{\overset{>0}{e^x}\,\overbrace{\left[x^2-4x+7\right]}^{>0}}{(x+1)^5}$$



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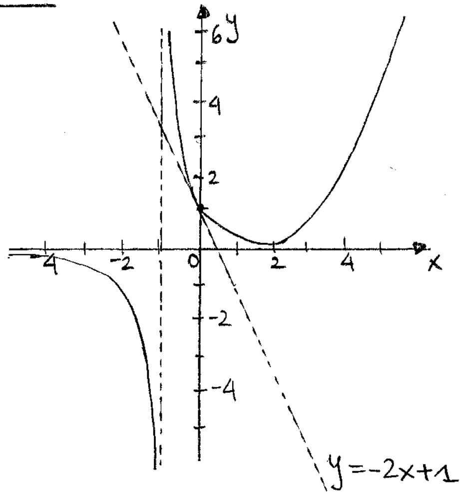
ii) $y = 1-2(x-0) \Rightarrow \underline{\underline{y=-2x+1}}$ ē l'eq. della retta tangente al grafico di f nel pt. $(0,1)$.

grafico qualitativo di f



iii) $\displaystyle\int_3^4 \frac{e^x}{f(x)}\,dx = \int_3^4 (x+1)^3\,dx =$

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$$\left[\frac{(x+1)^4}{4}\right]_3^4 = \frac{5^4}{4} - \frac{4^4}{4} = \frac{625-256}{4} = \underline{\underline{\frac{369}{4}}}.$$

■

6) $C_{n,2} = 15 \iff \dfrac{n(n-1)\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}} = 15 \iff n^2-n-30=0$

$\iff n_{1/2} = \dfrac{1\pm\sqrt{1+120}}{2} = \dfrac{1\pm 11}{2} = \begin{cases} 6 \\ -5 \end{cases} \Rightarrow \underline{\underline{n=6}}.$

■