Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
Esame Scritto di ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2011-2012 – Rovereto, 13 febbraio 2012

FILA A

1) i) Osserviamo che

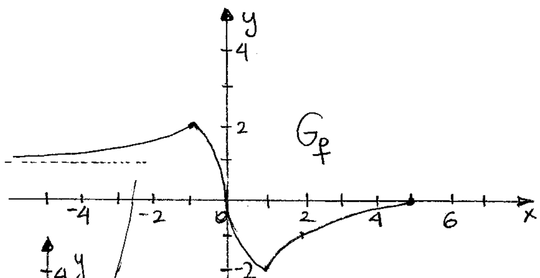
$\mathcal{A}$:	V	V	F	F
$\mathcal{B}$:	V	F	V	F
$\mathcal{A}$ e $\mathcal{B}$:	V	F	F	F
non $\mathcal{A}$:	F	F	V	V
non $\mathcal{B}$:	F	V	F	V
(non $\mathcal{A}$) o (non $\mathcal{B}$) :	F	V	V	V
non[(non $\mathcal{A}$) o (non $\mathcal{B}$)] :	V	F	F	F
$\mathcal{A}$ e $\mathcal{B}$ $\Leftrightarrow$ non[(non $\mathcal{A}$) o (non $\mathcal{B}$)] :	V	V	V	V

provando che " $\mathcal{A}$ e $\mathcal{B}$ $\Leftrightarrow$ non[(non $\mathcal{A}$) o (non $\mathcal{B}$)] " è una tautologia, cioè una proposizione vera per qualsiasi valore di verità degli elementi che la compongono.

ii) $\mathcal{A}$ = " $\exists x \in \mathbb{N} : x+2 > x^2$ " ; non $\mathcal{A}$ = " $\forall x \in \mathbb{N}$, $x+2 \leq x^2$ ".
Osserviamo che la proposizione $\mathcal{A}$ è vera; infatti si ha $x^2-x-2<0$ $\Leftrightarrow$ $x \in]-1,2[$, e $x = 1 \in]-1,2[$. Ne segue che la proposizione non $\mathcal{A}$ è falsa.

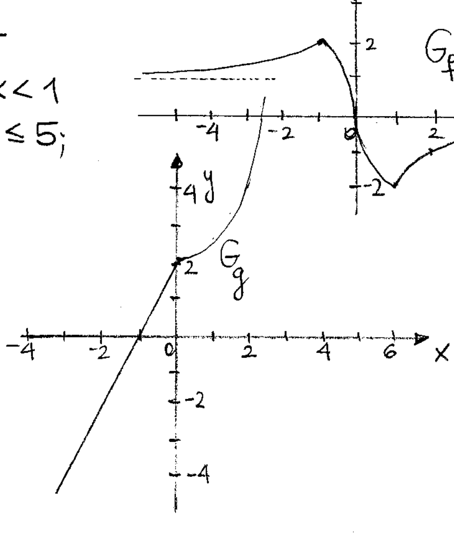
2) i) $f :]-\infty, 5] \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{1}{x^2}+1 & \text{se } x \leq -1 \\ -2\sqrt[3]{x} & \text{se } -1 < x < 1 \\ \sqrt{x-1}-2 & \text{se } 1 \leq x \leq 5; \end{cases}$$

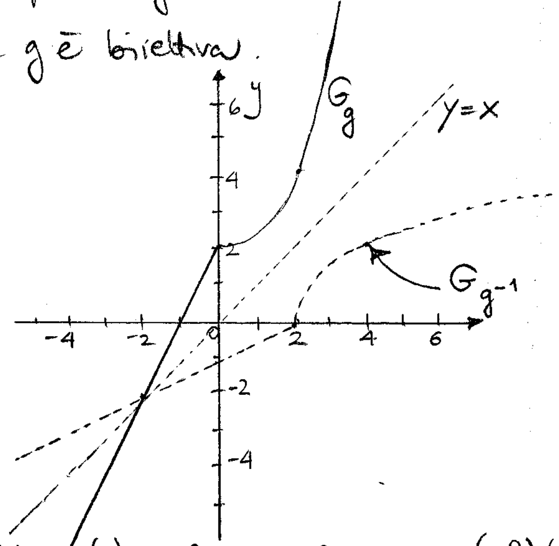


$g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} 2x+2 & \text{se } x<0 \\ \dfrac{x^2}{2}+2 & \text{se } 0 \leq x < 2 \\ 2^x & \text{se } x \geq 2. \end{cases}$$



ii) $g(\mathbb{R}) = \mathbb{R}$; quindi g è suriettiva. Osserviamo che g è iniettiva, e risulta che g è biiettiva.

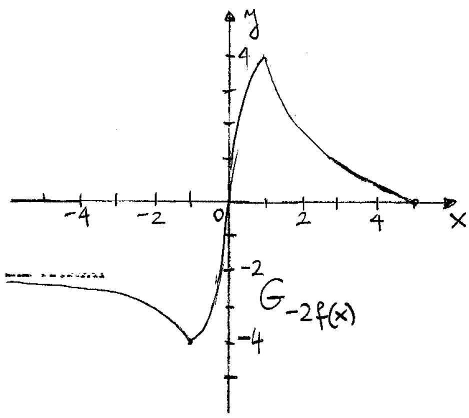


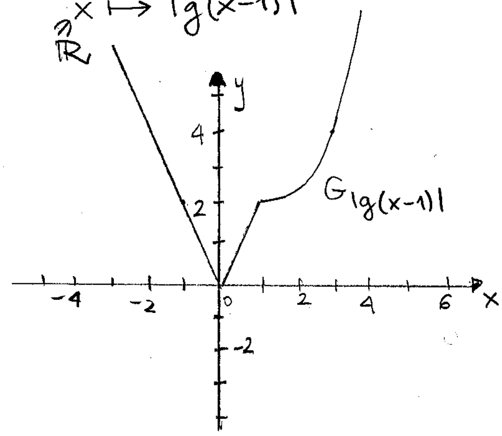
iii) $(f+g)(0) = f(0) + g(0) = 0 + 2 = \underline{\underline{2}}$ $\qquad (gf)(1) = g(1) \cdot f(1) = \frac{5}{2} \cdot (-2) = \underline{\underline{-5}}$

$(f \circ g)(0) = f(g(0)) = f(2) = \underline{\underline{-1}}$ $\qquad (g \circ f)(1) = g(f(1)) = g(-2) = \underline{\underline{-2}}$.

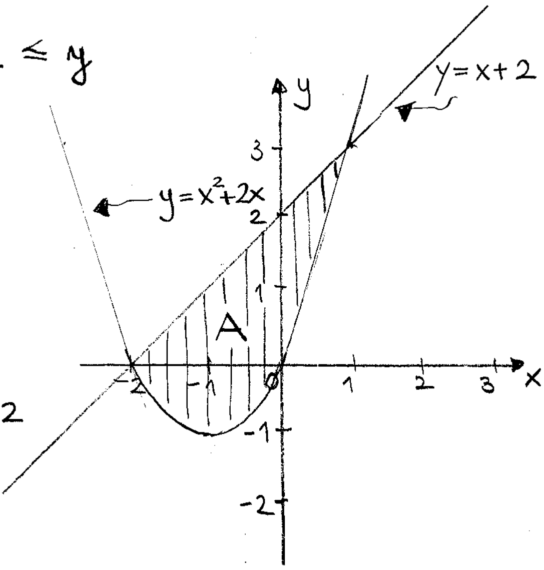
iv) $x \longmapsto -2f(x)$ $\qquad x \longmapsto |g(x-1)|$

$]-\infty, 5]$ $\qquad \mathbb{R}$





3) $\begin{cases} x^2 + 2x \leq y \\ y \leq x+2 \end{cases} \iff \begin{cases} (x+1)^2 - 1 \leq y \\ y \leq x+2 \end{cases}$

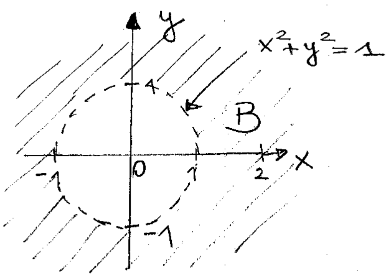


$x^2 + 2x = x + 2$

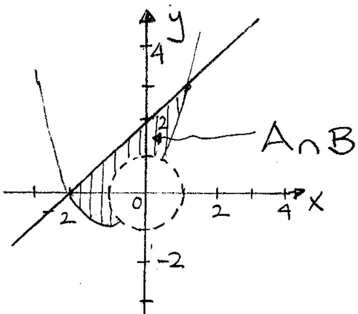
$\iff x^2 + x - 2 = 0$

$\iff x_{1/2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = \begin{cases} -2 \\ 1 \end{cases}$

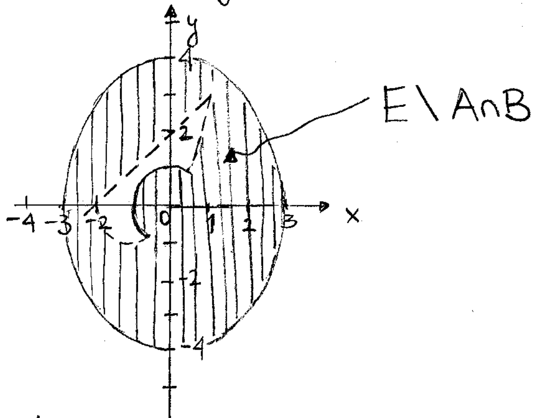
$B=\{(x,y)\in\mathbb{R}^2 : x^2+y^2>1\}$



i)



ii) $E=\{(x,y)\in\mathbb{R}^2 : 16x^2+9y^2\leq 144\}=\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{9}+\frac{y^2}{16}\leq 1\}$

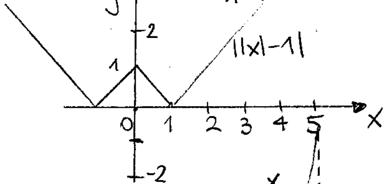


4) i) $\lim_{x\to+\infty}\frac{e^{-x}+3x^k}{1+4x^2}=\begin{cases}+\infty & \text{se } k=3\\ 3/4 & \text{se } k=2\\ 0 & \text{se } k=1.\end{cases}$

ii) $\int_1^2\frac{x^2+\sqrt{x}}{3x^3}\,dx=\frac{1}{3}\int_1^2\frac{1}{x}\,dx+\frac{1}{3}\int_1^2 x^{-5/2}\,dx=\frac{1}{3}\Big[\log|x|\Big]_1^2+\frac{1}{3}\left[\frac{x^{-3/2}}{-3/2}\right]_1^2$

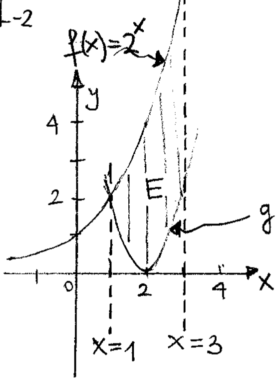
$=\underline{\underline{\frac{1}{3}\log 2-\frac{2}{9}\left(\frac{1}{\sqrt{8}}-1\right)}}.$

$\sum_{n=1}^{4}\int_{n-1}^{n+1}\big||x|-1\big|\,dx=\int_0^2\big||x|-1\big|dx+\int_1^3\big||x|-1\big|dx+\int_2^4\big||x|-1\big|dx+\int_3^5\big||x|-1\big|dx=$



$=1+2+4+6=\underline{\underline{13}}$

iii)

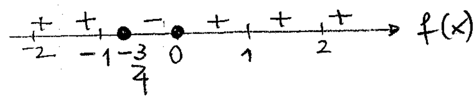


$\text{area}(E)=\int_1^3\big(f(x)-g(x)\big)dx=\int_1^3\left(2^x-2(x-2)^2\right)dx$

$=\left[\frac{2^x}{\log 2}-\frac{2(x-2)^3}{3}\right]_1^3=\left(\frac{8}{\log 2}-\frac{2}{3}\right)-\left(\frac{2}{\log 2}+\frac{2}{3}\right)$

$=\underline{\underline{\frac{6}{\log 2}-\frac{4}{3}}}$

5) i) $f(x)=\dfrac{4x^2+3x}{x^2+1}$ • $\operatorname{dom} f=\mathbb{R}=\,]-\infty,+\infty[$ nota: $x^2+1>0\ \forall x\in\mathbb{R}$



• $\lim\limits_{x\to-\infty} f(x)=4$

$\lim\limits_{x\to-\infty} f(x)=4$

$y=4$ è asintoto orizzontale per f, per $x\to-\infty$ (e per $x\to+\infty$)

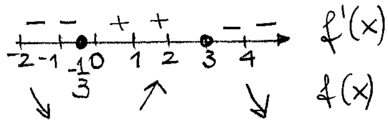
• f è continua su tutto $\mathbb{R}$, essendo somma e rapporto di funzioni continue e $x^2+1>0\ \forall x\in\mathbb{R}$.

• $\operatorname{dom} f'=\mathbb{R}$ $\quad f'(x)=\dfrac{(8x+3)(x^2+1)-(4x^2+3x)(2x)}{(x^2+1)^2}=\dfrac{8x+3x^2+3-6x^2}{(x^2+1)^2}$

$=\dfrac{-3x^2+8x+3}{(x^2+1)^2}$

$-3x^2+8x+3=0 \iff x_{1/2}=\dfrac{-8\pm\sqrt{64+36}}{-6}$

$=\dfrac{-8\pm10}{-6}\ \Big\langle\ \begin{matrix}-\frac{1}{3}\\ 3\end{matrix}$

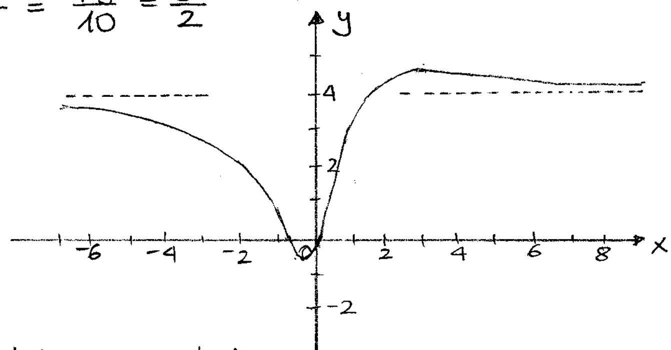


I pt. critici sono $x=-\frac{1}{3}$ e $x=3$

$f\left(-\frac{1}{3}\right)=\dfrac{4\cdot\frac{1}{9}-1}{\frac{1}{9}+1}=\dfrac{-5/9}{10/9}=-\frac{1}{2}$

$f(3)=\dfrac{4\cdot9+9}{10}=\dfrac{45}{10}=\dfrac{9}{2}$

grafico qualitativo di f



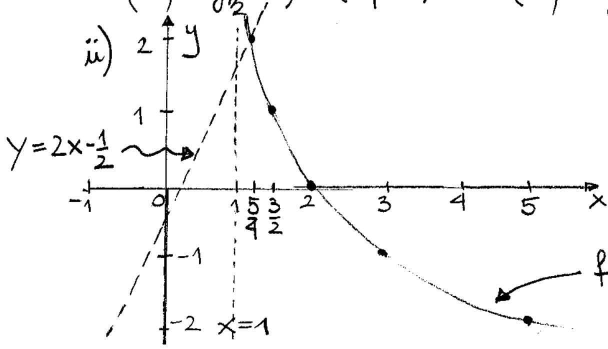
ii) $\left(-\frac{1}{3},-\frac{1}{2}\right)$, $\left(3,\frac{9}{2}\right)$ sono i pt. del grafico di f in cui la retta tangente al grafico è orizzontale.

iii) $y=-\frac{1}{2}$; $y=\frac{9}{2}$.

6) i) $\left(\frac{5}{4},\log_{\frac12}\frac{1}{4}\right)=\left(\frac{5}{4},2\right)$; $\left(\frac{3}{2},\log_{\frac12}\frac{1}{2}\right)=\left(\frac{3}{2},1\right)$; $\left(2,\log_{\frac12}1\right)=(2,0)$;

$\left(3,\log_{\frac12}2\right)=(3,-1)$; $\left(5,\log_{\frac12}4\right)=(5,-2)$;

ii)



$f(x)=\log_{\frac12}(x-1)$

$\operatorname{dom} f=\,]1,+\infty[$

iii) $S=\left[\frac{5}{4},+\infty\right[$.

FILA (B)

1) i) Osserviamo che

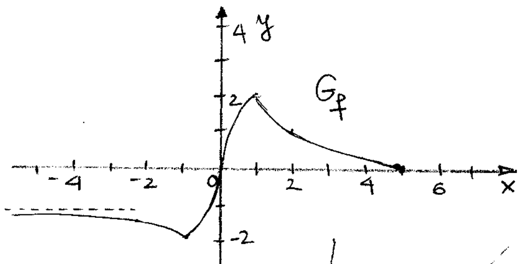
$\mathcal{A}$:	V	V	F	F
$\mathcal{B}$:	V	F	V	F
$\mathcal{A}$ o $\mathcal{B}$:	V	V	V	F
non $\mathcal{A}$:	F	F	V	V
non $\mathcal{B}$:	F	V	F	V
(non $\mathcal{A}$) e (non $\mathcal{B}$):	F	F	F	V
non[(non $\mathcal{A}$) e (non $\mathcal{B}$)]:	V	V	V	F
$\mathcal{A}$ o $\mathcal{B}$ $\Leftrightarrow$ non[(non $\mathcal{A}$) e (non $\mathcal{B}$)]:	V	V	V	V

provando che "$\mathcal{A}$ o $\mathcal{B}$ $\Leftrightarrow$ non[(non $\mathcal{A}$) e (non $\mathcal{B}$)]" è una tautologia, cioè una proposizione vera per qualsiasi valore di verità degli elementi che la compongono. □

ii) $\mathcal{A}$ = "$\forall x \in \mathbb{N},\ x+2<x^2$" ; non $\mathcal{A}$ = "$\exists x \in \mathbb{N}:\ x+2 \geq x^2$". Osserviamo che la proposizione non $\mathcal{A}$ é vera ; infatti si ha $x^2-x-2 \leq 0 \Leftrightarrow x \in [-1,2]$, e $x=1 \in [-1,2]$. Ne segue che la proposizione non $\mathcal{A}$ è vera, e quindi $\mathcal{A}$ è falsa. ■

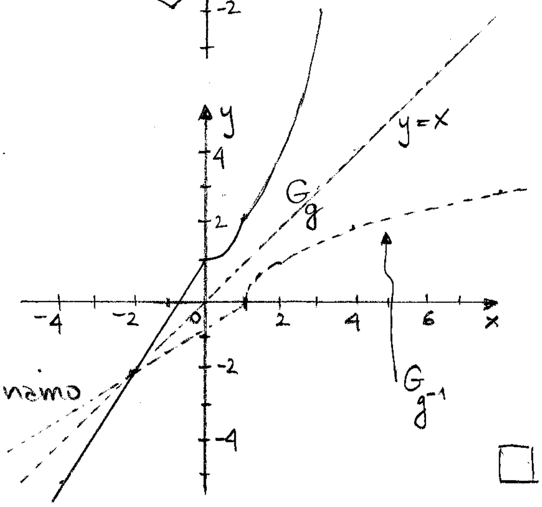
2) i) $f :]-\infty, 5] \to \mathbb{R}$

$$f(x)=\begin{cases} -\dfrac{1}{x^2}-1 & \text{se } x \leq -1 \\ 2\sqrt[3]{x} & \text{se } -1<x<1 \\ -\sqrt{x-1}+2 & \text{se } 1 \leq x \leq 5 ; \end{cases}$$



$g : \mathbb{R} \to \mathbb{R}$

$$g(x)=\begin{cases} \frac{3}{2}x+1 & \text{se } x<0 \\ x^2+1 & \text{se } 0 \leq x<1 \\ 2^x & \text{se } x \geq 1 . \end{cases}$$



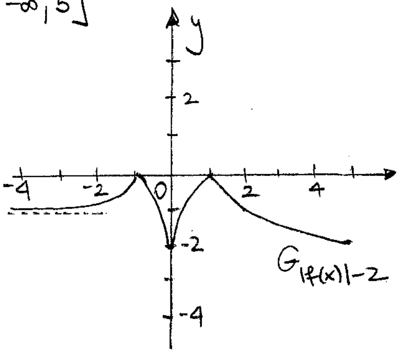
ii) $g(\mathbb{R})=\mathbb{R}$; quindi g è suriettiva. Osserviamo che g è iniettiva e risulta che g é biiettiva. □

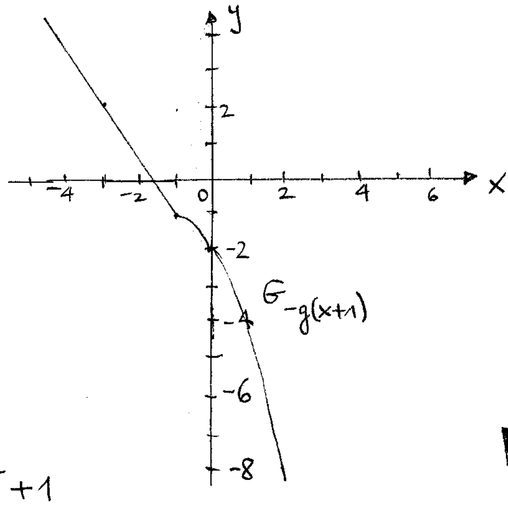
iii) $(f+g)(0) = f(0)+g(0) = 0+1 = \underline{\underline{1}}$ $\quad (gf)(1) = g(1)\cdot f(1) = 2\cdot 2 = \underline{\underline{4}}$

$(f\circ g)(0) = f(g(0)) = f(1) = \underline{\underline{2}}$ $\quad (g\circ f)(1) = g(f(1)) = g(2) = \underline{\underline{4}}.$

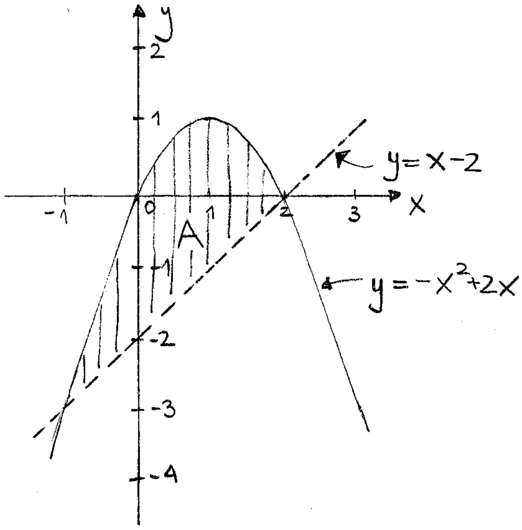
iv) $x \mapsto |f(x)|-2$ $\qquad x \mapsto -g(x+1)$

$\overset{\in}{]-\infty, 5]}$ $\qquad \overset{\in}{\mathbb{R}}$





3) $\begin{cases} y \leq -x^2+2x \\ y > x-2 \end{cases} \Leftrightarrow \begin{cases} y \leq -(x-1)^2+1 \\ y > x-2 \end{cases}$



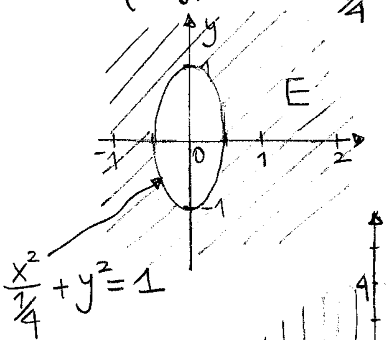
$-x^2+2x = x-2$

$\Leftrightarrow \quad x^2-x-2=0$

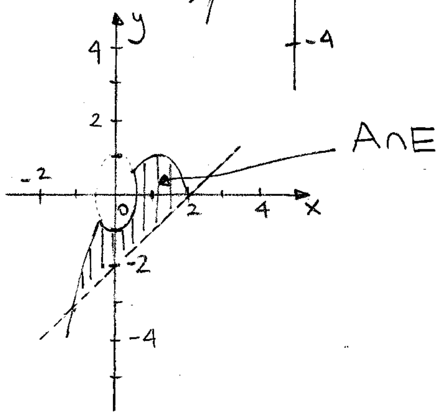
$\Leftrightarrow \quad x_{1/2} = \dfrac{1\pm\sqrt{1+8}}{2} = \dfrac{1\pm 3}{2} = \begin{cases} -1 \\ 2 \end{cases}$

$E = \{(x,y)\in\mathbb{R}^2 : 4x^2+y^2 \geq 1\} =$

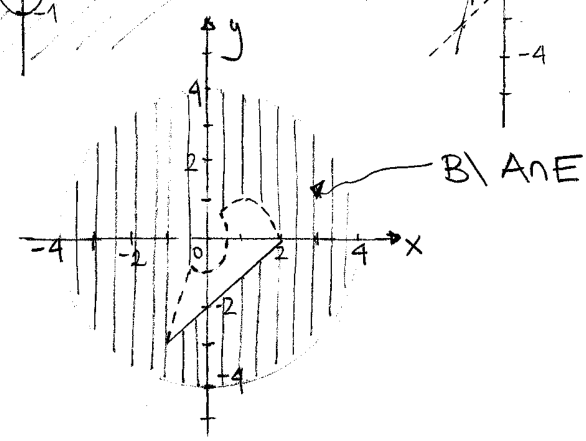
$= \{(x,y)\in\mathbb{R}^2 : \dfrac{x^2}{\frac{1}{4}}+y^2 \geq 1\}$



i)



ii)

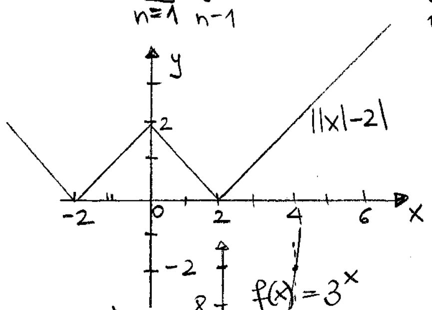


4) i) $\lim_{x\to+\infty} \dfrac{e^{-x}+3x^2}{1+4x^k} = \begin{cases} 0 & \text{se } k=3 \\ 3/4 & \text{se } k=2 \\ +\infty & \text{se } k=1. \end{cases}$ $\square$

ii) $\displaystyle\int_1^2 \frac{3x^2+\sqrt{x}}{2x^3}\,dx = \frac{3}{2}\int_1^2 \frac{1}{x}\,dx + \frac{1}{2}\int_1^2 x^{-5/2}\,dx = \frac{3}{2}\Big[\log|x|\Big]_1^2 + \frac{1}{2}\left[\frac{x^{-3/2}}{-3/2}\right]_1^2$

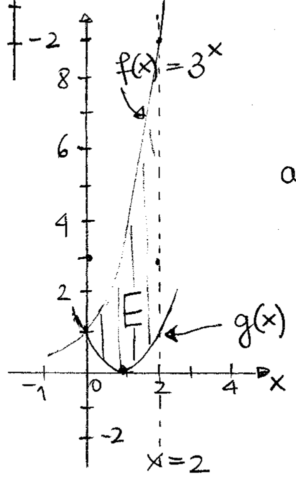
$\displaystyle = \underline{\underline{\frac{3}{2}\log 2 - \frac{1}{3}\left(\frac{1}{\sqrt{8}}-1\right)}}.$

$\displaystyle\sum_{n=1}^{4}\int_{n-1}^{n+1} ||x|-2|\,dx = \int_0^2 ||x|-2|\,dx + \int_1^3 ||x|-2|\,dx + \int_2^4 ||x|-2|\,dx + \int_3^5 ||x|-2|\,dx$



$= 2+1+2+4 = \underline{\underline{9}}$ $\square$

iii)



$\displaystyle \text{area}(E) = \int_0^2 (f(x)-g(x))\,dx = \int_0^2 \left(3^x-(x-1)^2\right)dx$

$\displaystyle = \left[\frac{3^x}{\log 3} - \frac{(x-1)^3}{3}\right]_0^2 = \left(\frac{9}{\log 3}-\frac{1}{3}\right) - \left(\frac{1}{\log 3}+\frac{1}{3}\right)$

$\displaystyle = \underline{\underline{\frac{8}{\log 3} - \frac{2}{3}}}$ ■

5) i) $f(x) = \dfrac{4x^2-3x}{x^2+1}$

- $\text{dom} f = \mathbb{R} =]-\infty,+\infty[$ nota: $x^2+1>0 \ \forall x\in\mathbb{R}$
- $\lim_{x\to-\infty} f(x) = 4$

  $\lim_{x\to+\infty} f(x) = 4$

  $+ \quad + \quad - \quad + \quad +$ $f(x)$ ($-1 \quad 0 \quad \tfrac{3}{4} \quad 1 \quad 2$)

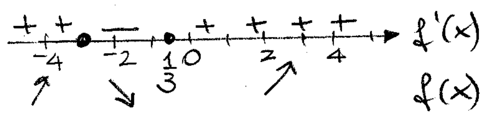
  $y=4$ è asintoto orizzontale per f per $x\to-\infty$ (e per $x\to+\infty$)

- f è continua su tutto $\mathbb{R}$, essendo somma e rapporto di funzioni continue su $\mathbb{R}$, e $x^2+1>0 \ \forall x\in\mathbb{R}$.
- $\text{dom} f' = \mathbb{R}$ $\quad f'(x) = \dfrac{(8x-3)(x^2+1)-(4x^2-3x)\cdot 2x}{(x^2+1)^2} = \dfrac{8x-3x^2-3+6x^2}{(x^2+1)^2}$

  $= \dfrac{3x^2+8x-3}{(x^2+1)^2}$

$3x^2+8x-3=0 \iff x_{1/2}=\dfrac{-8\pm\sqrt{64+36}}{6}=\dfrac{-8\pm 10}{6}=\begin{cases}-3\\ \frac{1}{3}\end{cases}$

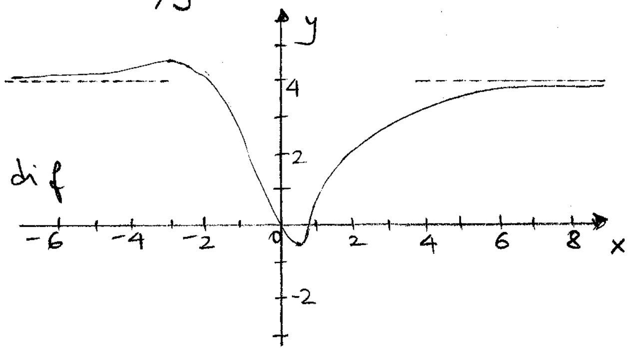
I punti critici sono $x=-3$ e $x=\frac{1}{3}$



$f(-3)=\dfrac{36+9}{10}=\dfrac{45}{10}=\dfrac{9}{2}$

$f\left(\frac{1}{3}\right)=\dfrac{4\cdot\frac{1}{9}-3\cdot\frac{1}{3}}{\frac{1}{9}+1}=\dfrac{-5/9}{10/9}=-\dfrac{1}{2}$

grafico qualitativo di f



ii) $\left(-3,\frac{9}{2}\right)$, $\left(\frac{1}{3},-\frac{1}{2}\right)$ sono i pt. del grafico di f in cui la retta tangente al grafico è orizzontale.

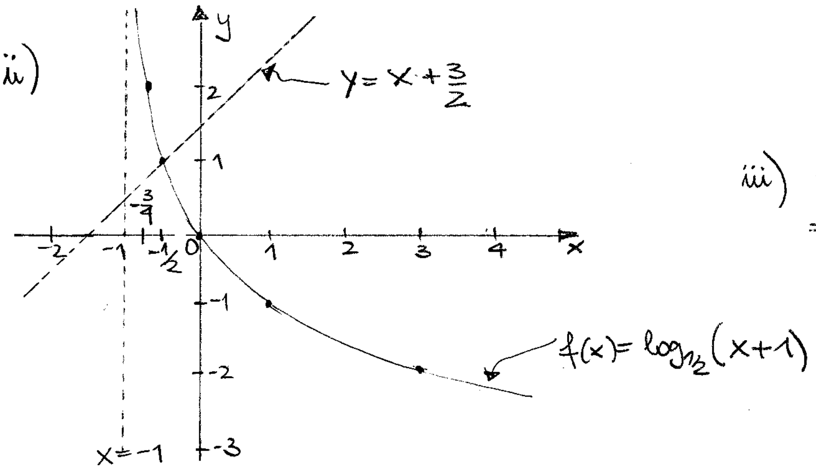
iii) $y=-\frac{1}{2}$; $y=\frac{9}{2}$.

6) i) $\mathrm{dom} f =]-1,+\infty[$

$\left(-\frac{3}{4},\log_{1/2}\frac{1}{4}\right)=\left(-\frac{3}{4},2\right)$; $\left(-\frac{1}{2},\log_{1/2}\frac{1}{2}\right)=\left(-\frac{1}{2},1\right)$; $\left(0,\log_{1/2}1\right)=(0,0)$;

$\left(1,\log_{1/2}2\right)=(1,-1)$; $\left(3,\log_{1/2}4\right)=(3,-2)$;

ii)



iii) $S=\left[-\frac{1}{2},+\infty\right[$.