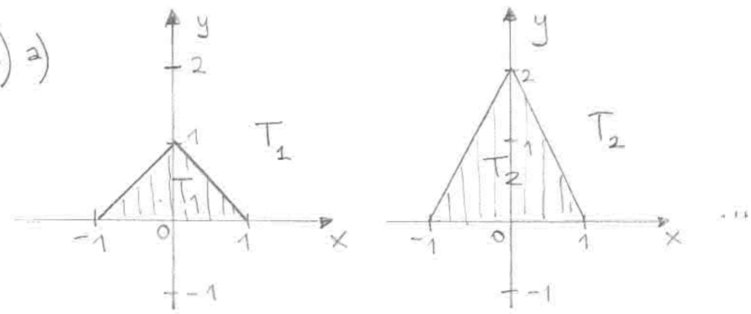
Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
VERIFICA SETTIMANALE DI ANALISI MATEMATICA (CON ELEMENTI DI ALGEBRA)
a.a. 2012-2013 — Rovereto, 3 - 7 dicembre 2012

1) i) $-\frac{a^2}{3}+\frac{a^4}{5}-\frac{a^6}{7}+\cdots+\frac{a^{20}}{21} = \underline{\underline{\sum_{n=1}^{10}(-1)^n\frac{a^{2n}}{2n+1}}}$;

$\frac{1}{2}+\frac{2}{3}+1+\frac{8}{5}+\cdots+\frac{512}{11} = \underline{\underline{\sum_{n=0}^{9}\frac{2^n}{n+2}}}$;

$\frac{1}{2}x-\frac{1}{3}x^2+\frac{1}{4}x^2-\frac{1}{6}x^3+\cdots+\frac{1}{20}x^{10}-\frac{1}{30}x^{11} = \underline{\underline{\sum_{n=1}^{10}\frac{1}{2n}x^n-\frac{1}{3n}x^{n+1}}}$. □

ii) a)



In generale, $\forall\ j \geq 1$

$\text{perimetro}(T_j) = 2+2\sqrt{1+j^2}$

$\text{area}(T_j) = \frac{2\cdot j}{2} = j$

Ne segue $\sum_{j=1}^{4}\text{perimetro}(T_j) = \sum_{j=1}^{4}\left(2+2\sqrt{1+j^2}\right)$

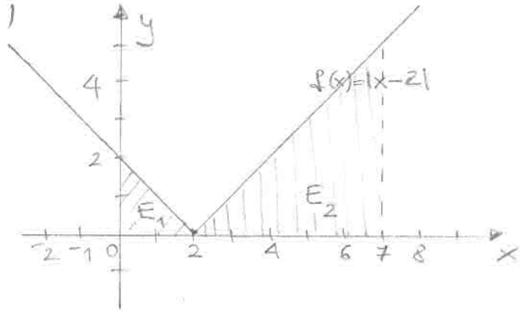
$= 2\cdot 4 + 2\sum_{j=1}^{4}\sqrt{1+j^2} = \underline{\underline{8+2\left(\sqrt{2}+\sqrt{5}+\sqrt{10}+\sqrt{17}\right)}}$

$\sum_{j=1}^{8}\text{area}(T_j) = \sum_{j=1}^{8} j = 1+2+3+\cdots+8 = \underline{\underline{36}}$;

b) $\sum_{n=1}^{5} f(n)$, dove $f(x) = \log\left(\frac{1}{e^x}\right) = \log\left(e^{-x}\right) = -x$; quindi

$\sum_{n=1}^{5} f(n) = -1-2-3-4-5 = \underline{\underline{-15}}$;

c) $\sum_{m=0}^{6}\int_m^{m+1}|x-2|\,dx =$

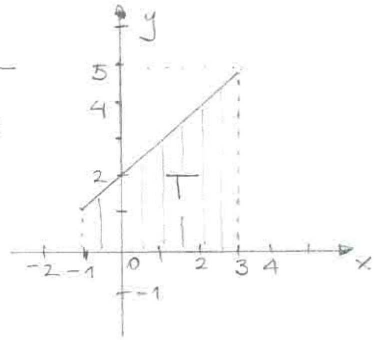


$\int_0^1|x-2|dx+\int_1^2|x-2|dx+\cdots+\int_6^7|x-2|dx$

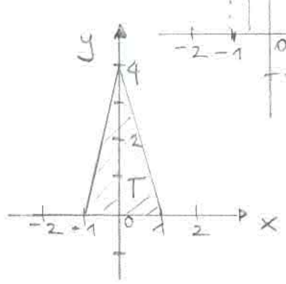
$= \int_0^7|x-2|\,dx = \text{area}(E_1)+\text{area}(E_2) = \frac{2\cdot 2}{2}+\frac{5\cdot 5}{2} = \underline{\underline{\frac{29}{2}}}$;

$\sum_{k=2}^{40}\left(\frac{1}{k(k-1)}-\frac{1}{k(k+1)}\right) = \left(\frac{1}{2\cdot 1}-\cancel{\frac{1}{2\cdot 3}}\right)+\left(\cancel{\frac{1}{3\cdot 2}}-\cancel{\frac{1}{3\cdot 4}}\right)+\left(\cancel{\frac{1}{4\cdot 3}}-\cancel{\frac{1}{4\cdot 5}}\right)+\cdots$

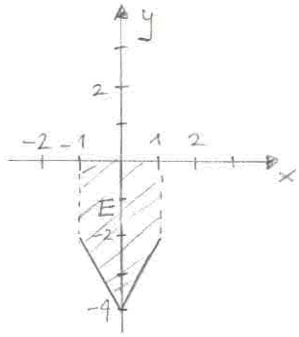
$+\left(\cancel{\frac{1}{39\cdot 38}}-\cancel{\frac{1}{39\cdot 40}}\right)+\left(\cancel{\frac{1}{40\cdot 39}}-\frac{1}{40\cdot 41}\right) = \frac{1}{2}-\frac{1}{40\cdot 41} = \underline{\underline{\frac{819}{1640}}}$. ■



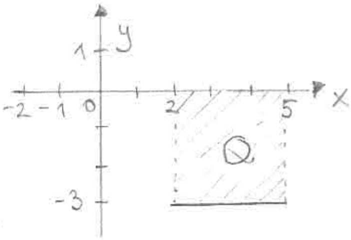
2) $\displaystyle\int_{-1}^{3}(x+2)\,dx = \text{area}(T) = \frac{(5+1)}{\cancel{2}}\cdot\cancel{4}^{\,2} = \underline{\underline{12}}\ ;$



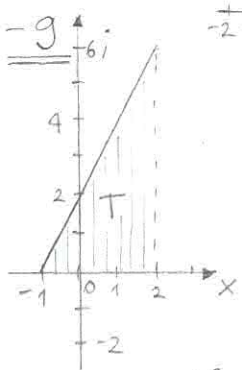
$\displaystyle\int_{-1}^{1}(-4|x|+4)\,dx = \text{area}(T) = \frac{\cancel{2}\cdot 4}{\cancel{2}} = \underline{\underline{4}}\ ;$



$\displaystyle\int_{-1}^{1}(2|x|-4)\,dx = -\text{area}(E) = -\left(2\cdot 2 + \frac{\cancel{2}\cdot 2}{\cancel{2}}\right) = \underline{\underline{-6}}\ ;$



$\displaystyle\int_{2}^{5}(-3)\,dx = -\text{area}(Q) = -3\cdot 3 = \underline{\underline{-9}}\ ;$



$\displaystyle\int_{-1}^{2}(2x+2)\,dx = \text{area}(T) = \frac{3\cdot\cancel{6}^{\,3}}{\cancel{2}} = \underline{\underline{9}}.$

3) i) $\displaystyle\int_{1}^{2}\left(x^{-2}+\frac{1}{x^3}\right)dx = \int_{1}^{2}\left(x^{-2}+x^{-3}\right)dx = \left[-x^{-1}+\frac{x^{-2}}{-2}\right]_{1}^{2}$

$\displaystyle = \left(-\frac{1}{2}-\frac{1}{8}\right)-\left(-1-\frac{1}{2}\right) = \underline{\underline{\frac{7}{8}}}\ ;$

$\displaystyle\int_{-3}^{-2}\left(e^x+\frac{2}{x}\right)dx = \left[e^x+2\log|x|\right]_{-3}^{-2} = \left(e^{-2}+2\log 2\right)-\left(e^{-3}+2\log 3\right)$

$\displaystyle = \underline{\underline{\frac{1}{e^2}-\frac{1}{e^3}+2\log\frac{2}{3}}}\ ;$

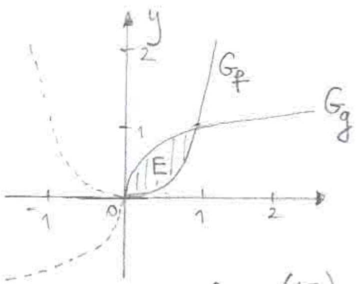
$\displaystyle\int_{1}^{4}\left(\frac{\sqrt{x}}{2}+x^2\right)dx = \left[\frac{x^{3/2}}{3}+\frac{x^3}{3}\right]_{1}^{4} = \left(\frac{8}{3}+\frac{64}{3}\right)-\left(\frac{1}{3}+\frac{1}{3}\right) = 2+\frac{64}{3} = \underline{\underline{\frac{70}{3}}}\ ;$

ii) $\displaystyle\int_{-1}^{3}\left(3^x+\sqrt[3]{x}\right)dx = \left[\frac{3^x}{\log 3}+\frac{3}{4}x^{4/3}\right]_{-1}^{3} = \underline{\underline{\left(\frac{27}{\log 3}+\frac{9\sqrt[3]{3}}{4}\right)-\left(\frac{1}{3\log 3}+\frac{3}{4}\right)}}\ ;$

$\displaystyle\int_{1}^{2}\frac{2x^3-4x^2}{x}\,dx = \int_{1}^{2}(2x^2-4x)\,dx = \left[2\frac{x^3}{3}-2x^2\right]_{1}^{2} = \left(\frac{16}{3}-8\right)-\left(\frac{2}{3}-2\right) = \underline{\underline{-\frac{4}{3}}}\ ;$

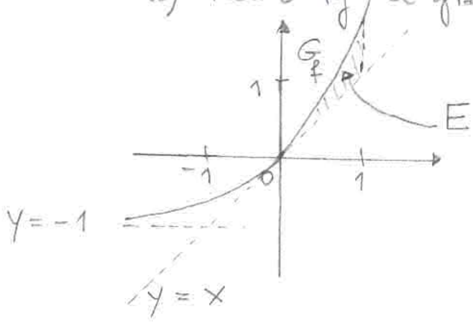
$\displaystyle\int_{0}^{1}\frac{e^{2x}-1}{e^x+1}\,dx = \int_{0}^{1}\frac{(e^x-1)\cancel{(e^x+1)}}{\cancel{e^x+1}}\,dx = \left[e^x-x\right]_{0}^{1} = (e-1)-(1) = \underline{\underline{e-2}}.$

4) a) i)



$$\text{area}(E) = \int_0^1 \left(\sqrt[3]{x} - x^4\right)dx = \left[\frac{3}{4}x^{4/3} - \frac{x^5}{5}\right]_0^1 = \frac{3}{4} - \frac{1}{5} = \underline{\underline{\frac{11}{20}}};$$

ii) Retta tg al grafico di $f(x) = e^x - 1$ nel punto $(0,0)$ è $y = x$



$$\text{area}(E) = \int_0^1 (e^x - 1 - x)\,dx = \left[e^x - x - \frac{x^2}{2}\right]_0^1 =$$

$$= \left(e - 1 - \tfrac{1}{2}\right) - (1) = \underline{\underline{e - \frac{5}{2}}};$$

b) $F(x) = \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\log x - \frac{x^3}{9} - \frac{x^2}{4}$ è una funzione derivabile su $]0,+\infty[$ (essendo prodotto e somma di funzioni derivabili su $]0,+\infty[$). Inoltre

$$\underline{\underline{F'(x)}} = \left(\frac{\cancel{3}x^2}{\cancel{3}} + \frac{\cancel{2}x}{\cancel{2}}\right)\log x + \left(\frac{x^3}{3} + \frac{x^2}{2}\right)\cdot\frac{1}{x} - \frac{\cancel{3}x^2}{\cancel{9}_3} - \frac{\cancel{2}x}{\cancel{4}_2} = \underline{\underline{(x^2+x)\log x = f(x)}}$$

su $]0,+\infty[$. Quindi $F(x)$ è una primitiva di $f(x)$. Abbiamo quindi

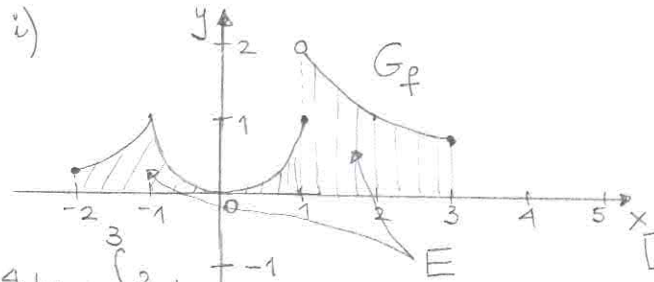
$$\int_1^e f(x)\,dx = \Big[F(x)\Big]_1^e = \left[\left(\frac{x^3}{3} + \frac{x^2}{2}\right)\log x - \frac{x^3}{9} - \frac{x^2}{4}\right]_1^e = \left(\frac{e^3}{3} + \frac{e^2}{2} - \frac{e^3}{9} - \frac{e^2}{4}\right) - \left(-\frac{1}{9} - \frac{1}{4}\right.$$

$$= \underline{\underline{\frac{2}{9}e^3 + \frac{e^2}{4} + \frac{13}{36}}}.$$

5) $f : [-2,3] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{se } -2 \le x \le -1 \\ x^4 & \text{se } -1 < x \le 1 \\ 2/x & \text{se } 1 < x \le 3 \end{cases}$$

i)



ii)
$$\int_{-2}^{3} f(x)\,dx = \int_{-2}^{-1} \frac{1}{x^2}\,dx + \int_{-1}^{1} x^4\,dx + \int_1^3 \frac{2}{x}\,dx$$

$$= \Big[x^{-1}\Big]_{-2}^{-1} + \left[\frac{x^5}{5}\right]_{-1}^{1} + 2\Big[\log|x|\Big]_1^3 =$$

$$= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{5} + \tfrac{1}{5}\right) + 2\log 3 = \tfrac{1}{2} + \tfrac{2}{5} + 2\log 3 = \underline{\underline{\frac{9}{10} + \log 9}}.$$

iii) $\int_{-2}^{3} f(x)\,dx = \text{area}(E)$.

iv) $x = -1$ pt. di massimo locale per f; $x = -2$, $x = 0$, $x = 3$ pt. di minimo locale per f.

6) i) $\boxed{g(x) = \frac{1}{x(x-2)}}$ a) • $\text{dom}\, g = \mathbb{R}\setminus\{0,2\} =]-\infty,0[\cup]0,2[\cup]2,+\infty[$

• segno di g

$$\begin{array}{ccccccccc} + & + & 0 & - & - & - & 0 & + & + \\ -1 & & 0 & & 1 & & 2 & & 3 \end{array} \quad x^2-2x$$

• Comportamento agli estremi del dom g :

$\lim_{x\to-\infty} g(x) = 0$ — $y=0$ asintoto orizzontale per g, per $x \to -\infty$;

$\lim_{x\to 0^-} g(x) = +\infty$ — $x=0$ asintoto verticale per g

$\lim_{x\to 0^+} g(x) = -\infty$ — "

$\lim_{x\to 2^-} g(x) = -\infty$ — $x=2$ asintoto verticale per g

$\lim_{x\to 2^+} g(x) = +\infty$ — "

$\lim_{x\to+\infty} g(x) = 0$ — $y=0$ asintoto orizzontale per g, per $x\to+\infty$.

• g è continua su $\mathbb{R}\setminus\{0,2\}$.

• $\text{dom}\, g' = \mathbb{R}\setminus\{0,2\}$ $\quad g'(x) = \dfrac{-2x+2}{(x^2-2x)^2}$

$g'(x)=0 \iff x=1$ pt. critico per g

$$\begin{array}{cccccccccl} + & + & + & + & \bullet & - & - & - & & -2x+2 \\ -1 & & 0 & & 1 & & 2 & & & \\ + & + & 0 & + & + & 0 & + & + & & (x^2-2x)^2 \\ -1 & & 0 & & 1 & & 2 & & & \\ + & + & 0 & + & \bullet & - & 0 & - & - & g'(x) \\ -1 & & 0 & & 1 & & 2 & & & \\ \nearrow & & \nearrow & & \searrow & & \searrow & & & g(x) \end{array}$$

$x=1$ pt. di max. loc. stretto per g ; $\underline{\underline{g(1) = -1}}$.

• $\text{dom}\, g'' = \mathbb{R}\setminus\{0,2\}$

$$g''(x) = \frac{-2(x^2-2x)^{\cancel{2}} - (-2x+2)\cdot 2\cancel{(x^2-2x)}(2x-2)}{(x^2-2x)^{\cancel{4}\,3}}$$

$$= \frac{-2x^2+4x+8x^2-16x+8}{(x^2-2x)^3} = \frac{6x^2-12x+8}{(x^2-2x)^3}$$

$$\begin{array}{cccccccccl} + & + & + & + & + & + & & 6x^2-12x+8 \\ -1 & 0 & 1 & 2 & 3 & & & \\ + & + & 0 & - & - & 0 & + & + & (x^2-2x)^3 \\ & & 0 & & & 2 & & & \\ + & + & 0 & - & - & 0 & + & + & g''(x) \\ -1 & & 0 & 1 & & 2 & 3 & & \\ \cup & & & \cap & & & \cup & & g(x) \end{array}$$

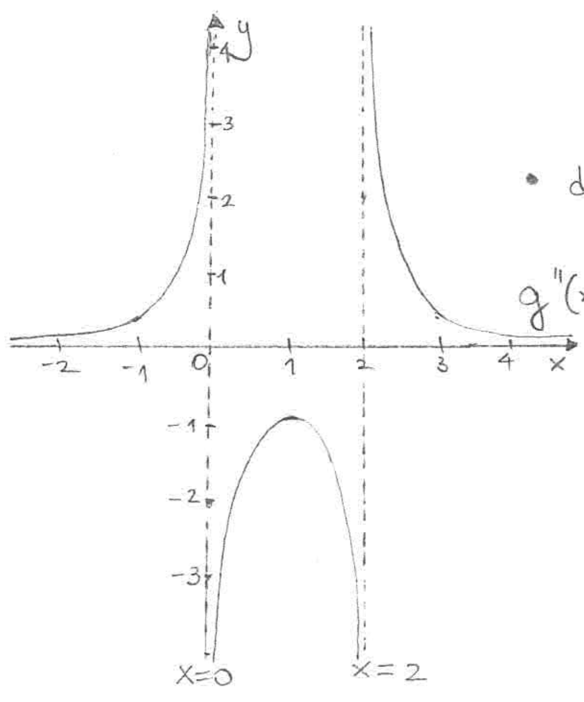


grafico qualitativo di g

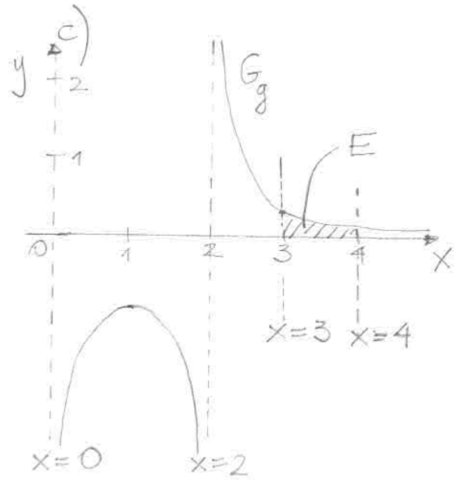
$\square$

b) $G(x)=\frac{1}{2}\left(\log(x-2)-\log x\right)$ è una funzione derivabile (composizione di funzioni derivabili e somma di funzioni derivabili) su $]2,+\infty[$;

inoltre $G'(x)=\frac{1}{2}\left[\frac{1}{x-2}-\frac{1}{x}\right]=\frac{1}{2}\,\frac{\cancel{x}-\cancel{x}+2}{x(x-2)}=\frac{1}{x(x-2)}=g(x)$

su $]2,+\infty[$. Quindi G è una funzione primitiva di g su $]2,+\infty[$. $\square$

c)



$$\text{area}(E)=\int_3^4 g(x)\,dx=\Big[G(x)\Big]_3^4=G(4)-G(3)$$

$$=\frac{1}{2}\left(\log 2-\log 4\right)-\frac{1}{2}\left(\cancel{\log 1}-\log 3\right)$$

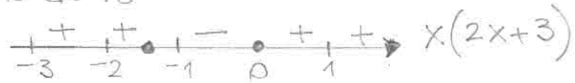
$$=\frac{1}{2}\left(\log 2-\log 4+\log 3\right)=\frac{1}{2}\log\frac{6}{4}=$$

$$=\frac{1}{2}\log\frac{3}{2}\,.$$

■

ii) $\boxed{h(x)=(2x^2+3x)\,e^x}$ a) • dom $h=\mathbb{R}=\,]-\infty,+\infty[$

• segno di h



(nota: $e^x>0\ \forall x\in\mathbb{R}$)

• Comportamento agli estremi del dom h:

$$\lim_{x\to-\infty}\frac{2x^2+3x}{e^{-x}}=\lim_{x\to+\infty}\frac{2x^2-3x}{e^x}=0$$

$y=0$ è asintoto orizzontale per h, per $x\to-\infty$

$$\lim_{x\to+\infty}(2x^2+3x)e^x=+\infty\,.$$

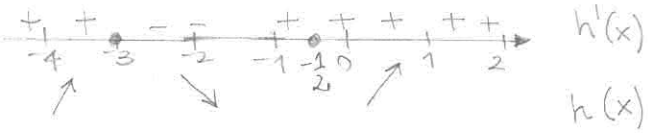
• h è continua su $\mathbb{R}$ (essendo prodotto di funzioni continue su $\mathbb{R}$).

• dom $h'=\mathbb{R}$ $\quad h'(x)=(4x+3)e^x+(2x^2+3x)e^x$

$$=(2x^2+7x+3)e^x$$

$h'(x)=0 \iff 2x^2+7x+3=0 \qquad x_{1/2}=\frac{-7\pm\sqrt{49-24}}{4}=\frac{-7\pm5}{4}$

$x_{1/2}=\begin{cases}-3\\-\frac{1}{2}\end{cases}$ pt. critici per h



$x = -3$ pt. di max loc. stretto per h

~ 0.4

$h(-3) = (18-9)e^{-3} = \underline{\underline{\dfrac{9}{e^3}}}$;

$x = -\frac{1}{2}$ pt. di min. loc. stretto per h

$h\left(-\frac{1}{2}\right) = \left(\cancel{2}\cdot\frac{1}{\cancel{4}_2} - \frac{3}{2}\right)e^{-1/2} = \underline{\underline{-\dfrac{1}{\sqrt{e}}}}$;

~ -0.6

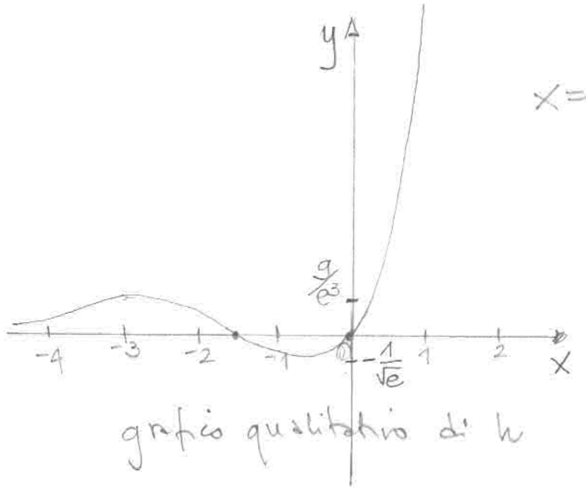


grafico qualitativo di h

b) $H(x) = (2x^2 - x + 1)e^x$ è una funzione derivabile su $\mathbb{R}$, essendo prodotto di funzioni derivabili su $\mathbb{R}$; inoltre

$H'(x) = (4x-1)e^x + (2x^2 - x + 1)e^x =$
$= (2x^2 + 3x)e^x = h(x) \quad \forall x \in \mathbb{R}$,

quindi $H(x)$ è una primitiva di $h(x)$ su $\mathbb{R}$. □

c) $\displaystyle\int_0^1 h(x)\,dx = \Big[H(x)\Big]_0^1$
$= H(1) - H(0) = \underline{\underline{2e - 1}}.$ ■