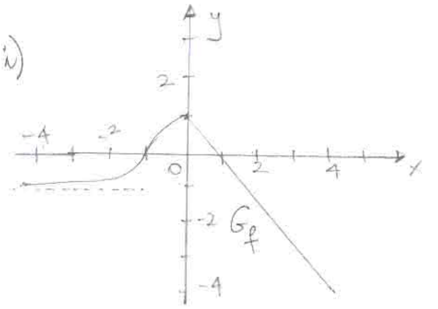
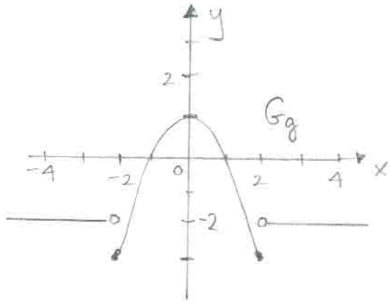
Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia cognitiva
CdL in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
ESAME SCRITTO DI ANALISI MATEMATICA (CON ELEMENTI DI ALGEBR
a.a. 2012-2013 — Rovereto, 21 gennaio 2013

FILA B

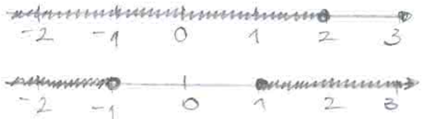
1) $f(x)=\begin{cases} 1/x^2-1 & \text{se } x<-1 \\ \log_2(x+2) & \text{se } -1\le x\le 0 \\ -x+1 & \text{se } x>0\,; \end{cases}$ $\qquad g(x)=\begin{cases} -x^2+1 & \text{se } x\in[-2,2] \\ -2 & \text{se } x\in\mathbb{R}\setminus[-2,2]. \end{cases}$

i)



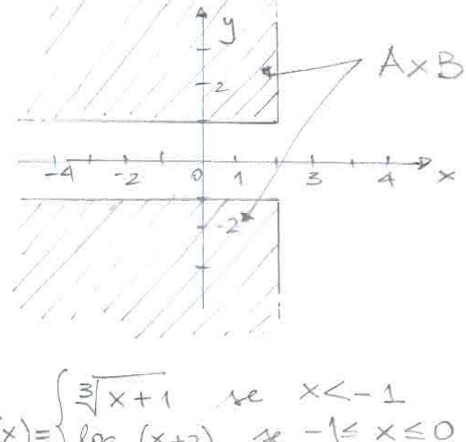


ii) $A=\{x\in\mathbb{R}: f(x)\ge -1\} =]-\infty,2]$ $\qquad B=\{x\in\mathbb{R}: g(x)\le 0\} =]-\infty,-1]\cup[1,+\infty[$

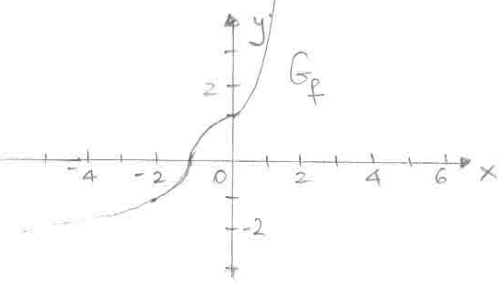


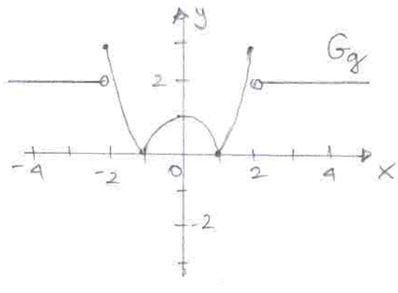
iii) $A\cap B =]-\infty,-1]\cup[1,2]$ $\qquad A\cup B=\mathbb{R}$ $\qquad A\setminus B =]-1,1[$.

iv)



2) i) $f(x)=\begin{cases} \sqrt[3]{x+1} & \text{se } x<-1 \\ \log_2(x+2) & \text{se } -1\le x\le 0 \\ 4^x & \text{se } x>0\,; \end{cases}$ $\qquad g(x)=\begin{cases} |-x^2+1| & \text{se } x\in[-2,2] \\ 2 & \text{se } x\in\mathbb{R}\setminus[-2,2]. \end{cases}$



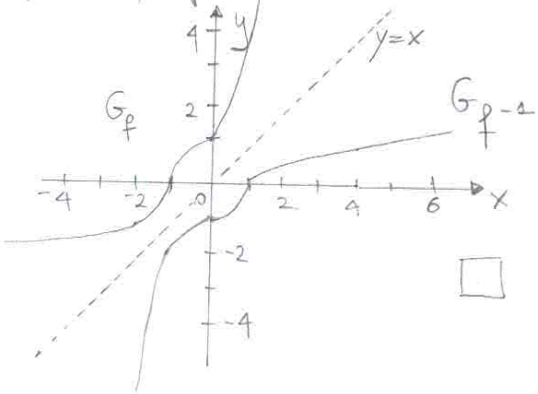


ii) $\forall\, k \in \mathbb{R}$, $\exists!$ soluzione dell'eq. $f(x)=k$. ☐

iii) $g(\mathbb{R}) = \underline{\underline{[0,3]}}$; poiché $g(\mathbb{R}) \subsetneq \mathbb{R}$, g non è suriettiva.
Inoltre, g non è iniettiva: infatti, basta prendere, per esempio, $x_1 = 3 \neq x_2 = 4$ e $g(x_1) = g(x_2) = 2$. ☐

iv) $(g\circ f)(-1) \doteq g(f(-1)) = g(0) = \underline{\underline{1}}$;
$(fg)(2) \doteq f(2)\,g(2) = 4^2 \cdot 3 = 16 \cdot 3 = \underline{\underline{48}}$;
$(f+g)\left(\tfrac{3}{2}\right) \doteq f\left(\tfrac{3}{2}\right) + g\left(\tfrac{3}{2}\right) = 4^{3/2} + 5/4 = 8 + 5/4 = \underline{\underline{\tfrac{37}{4}}}$ ☐

v) $f(\mathbb{R}) = \mathbb{R}$; f è iniettiva e suriettiva ; $\exists\, f^{-1} : \mathbb{R} \to \mathbb{R}$



☐

vi) $\max_{[-2,1]} f = f(1) = \underline{\underline{4}}$
$\underline{\underline{x=1}}$ pt. di max.

$\min_{[-2,1]} f = f(-2) = \underline{\underline{-1}}$
$\underline{\underline{x=-2}}$ pt. di min

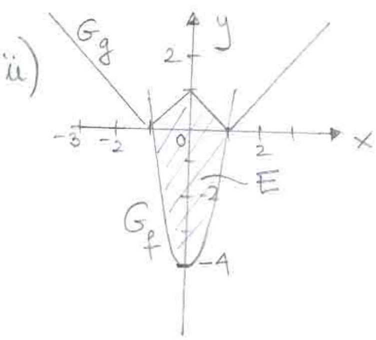
3) i) • $\int_0^1 \left(\frac{e^x}{e^x+2} + x^{1/4} \right) dx = \left[\log(e^x+2) + \frac{x^{5/4}}{5/4} \right]_0^1 = \log(e+2) + \frac{4}{5} - \log 3$

$= \underline{\underline{\log\left(\frac{e+2}{3}\right) + \frac{4}{5}}}.$

• $\lim_{x\to 0^-} \frac{e^x - 1}{x^2} = \lim_{x \to 0^-} \underbrace{\frac{e^x-1}{x}}_{\text{limite notevole } \searrow 1} \cdot \frac{1}{x} = \underline{\underline{-\infty}}$.

• $\lim_{x\to 0} \frac{e^{4x}-1}{x} = \lim_{x\to 0} \overbrace{\frac{e^{4x}-1}{4x}}^{\to 1} \cdot 4 = \underline{\underline{4}}$. ☐

ii)



$\text{area}(E) = \int_{-1}^{1} (g(x)-f(x))\,dx = 2\int_0^1 (g(x)-f(x))\,dx$
$= \cancel{2} \cdot \frac{1 \cdot 1}{\cancel{2}} - 2\int_0^1 f(x)\,dx = 1 - 2\int_0^1 (4x^2 - 4)\,dx$
$= 1 - 2\left[4\frac{x^3}{3} - 4x \right]_0^1 = 1 + \frac{16}{3} = \underline{\underline{\frac{19}{3}}}$ ☐

iii) • $\log(2-x^2) + \log_{1/e}(|x|) < 0 \iff \log(2-x^2) - \log(|x|) < 0$

$$\iff \log(2-x^2) < \log(|x|) \underset{\log x \uparrow}{\iff} \begin{cases} 2-x^2 > 0 \\ x \neq 0 \\ 2-x^2 < |x| \end{cases}$$

$$\iff \begin{cases} x^2 < 2 \\ x \neq 0 \\ x^2 + |x| - 2 > 0 \end{cases} \iff \begin{cases} -\sqrt{2} < x < 0 \\ x^2 - x - 2 > 0 \end{cases} \text{ o } \begin{cases} 0 < x < \sqrt{2} \\ x^2 + x - 2 > 0 \end{cases}$$

$$\iff \begin{cases} -\sqrt{2} < x < 0 \\ x < -1 \text{ o } x > 2 \end{cases} \text{ o } \begin{cases} 0 < x < \sqrt{2} \\ x < -2 \text{ o } x > 1 \end{cases}$$

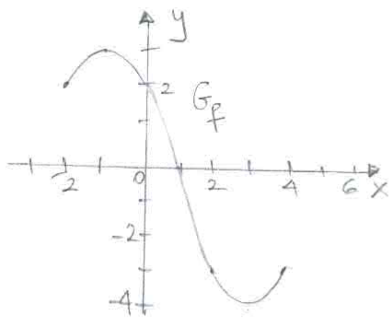
$S = \left]-\sqrt{2}, -1\right[\cup \left]1, \sqrt{2}\right[.$

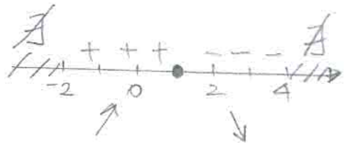
• $\log_2 32 < 5^{|x-1|-1} \iff 5 < 5^{|x-1|-1}$

$\underset{5^x \uparrow}{\iff} 1 < |x-1| - 1$

$\iff |x-1| > 2 \iff x-1 < -2 \text{ o } x-1 > 2$

$\Rightarrow S = \left]-\infty, -1\right[\cup \left]3, +\infty\right[$

4) 

i) 

$f = F'$ (teorema fond. del calcolo integrale)

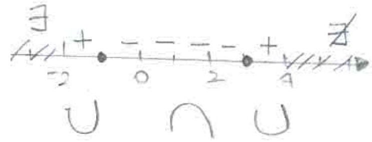
F

Si ha che F ristretta a $[-2, 1]$ è crescente

F ristretta a $[1, 4]$ è decrescente.

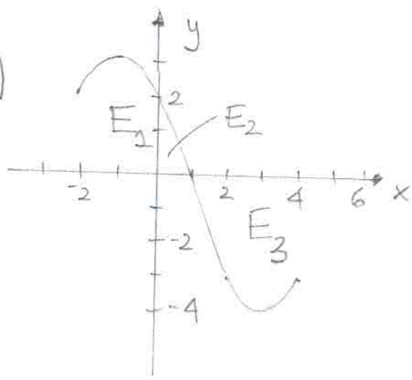
ii) $x = -2$, $x = 4$ sono pt. di minimo loc. di F,

$x = 1$ è pt. di massimo loc. di F.

iii) 

$f' = F''$

F

iv) 

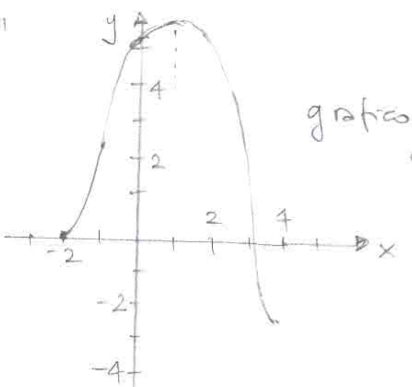


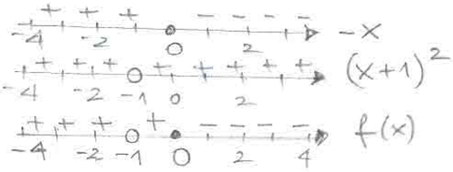
grafico qualitativo di F

$F(0) = \text{area}(E_1) > 2 \cdot 2 = 4$.

5) i) $f(x) = -\dfrac{x}{(x+1)^2}$

• $\operatorname{dom} f = \mathbb{R}\setminus\{-1\} =]-\infty,-1[\ \cup\ ]-1,+\infty[$.

• segno f



• $\lim_{x\to-\infty} f(x) = 0$ $\quad y=0$ asintoto orizzontale per f, per $x\to-\infty$;

$\lim_{x\to-1^-} f(x) = +\infty$ $\quad x=-1$ asintoto verticale per f

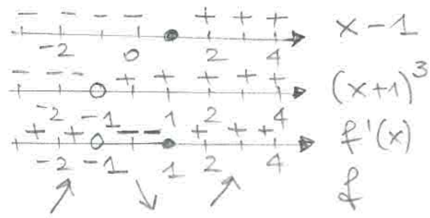
$\lim_{x\to-1^+} f(x) = +\infty$ $\quad x=-1$ "

$\lim_{x\to+\infty} f(x) = 0$ $\quad y=0$ asintoto orizzontale per f, per $x\to+\infty$.

• f è continua su $\mathbb{R}\setminus\{-1\}$.

• $\operatorname{dom} f' = \mathbb{R}\setminus\{-1\}$ $\quad f'(x) = \dfrac{-(x+1)^{\cancel{2}} + x\cdot 2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}} = \dfrac{x-1}{(x+1)^3}$

$f'(x)=0 \iff \underline{x=1 \text{ pt. critico per } f}$

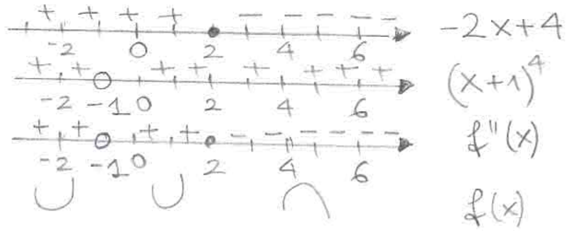


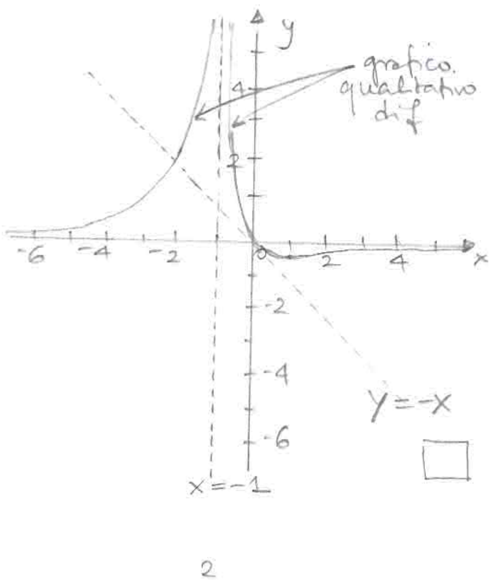
$x=1$ pt. di min. loc. stretto per f

$f(1) = -\dfrac{1}{2^2} = \underline{\underline{-\dfrac{1}{4}}}$.

• $\operatorname{dom} f'' = \mathbb{R}\setminus\{-1\}$ $\quad f''(x) = \dfrac{(x+1)^{\cancel{3}} - (x-1)\cdot 3\cancel{(x+1)^2}}{(x+1)^{\cancel{6}\,4}} =$

$= \dfrac{x+1-3x+3}{(x+1)^4} = \dfrac{-2x+4}{(x+1)^4}$



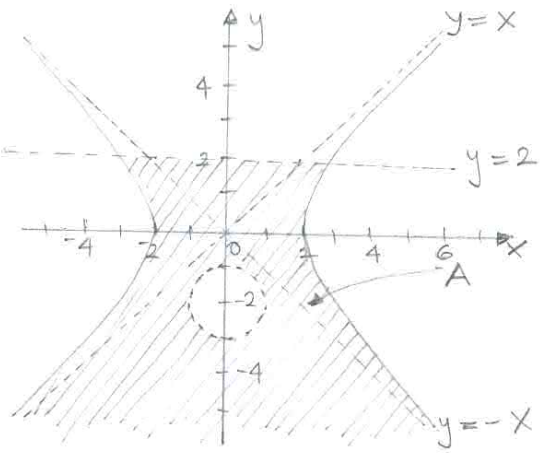


ii) Eq. retta tg. al grafico di f nel punto $(0,0)$

$y = f(0) + f'(0)x \Rightarrow \underline{y=-x}$.

iii) $\displaystyle\int_1^2 \frac{f'(x)}{f(x)}\,dx = \Big[\log|f(x)|\Big]_1^2 = \log\left(\frac{2}{9}\right) - \log\left(\frac{1}{4}\right) = \underline{\underline{\log\frac{8}{9}}}$.

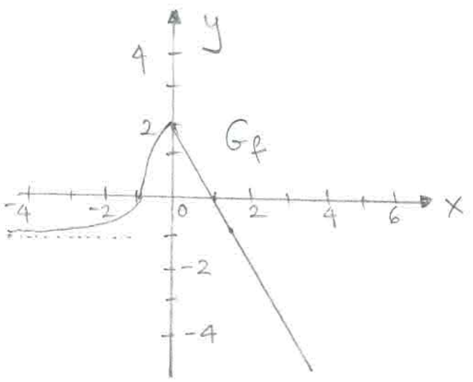
6) $\begin{cases} x^2 - y^2 \leq 4 \\ x^2 + y^2 + 4y + 3 > 0 \\ y < 2 \end{cases} \iff \begin{cases} \frac{x^2}{4} - \frac{y^2}{4} \leq 1 \\ x^2 + (y+2)^2 > 1 \\ y < 2 \end{cases}$

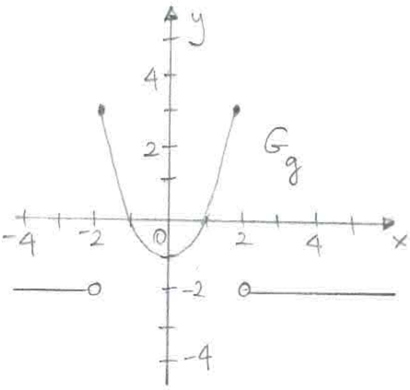


FILA C

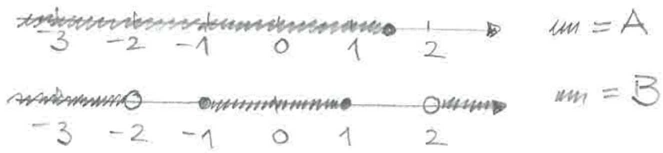
1) $f(x) = \begin{cases} 1/x^2 - 1 & \text{se } x < -1 \\ 2\log_2(x+2) & \text{se } -1 \leq x \leq 0 \\ -2x + 2 & \text{se } x > 0 \end{cases}$ $\qquad g(x) = \begin{cases} x^2 - 1 & \text{se } x \in [-2,2] \\ -2 & \text{se } x \in \mathbb{R} \setminus [-2,2]. \end{cases}$

i)





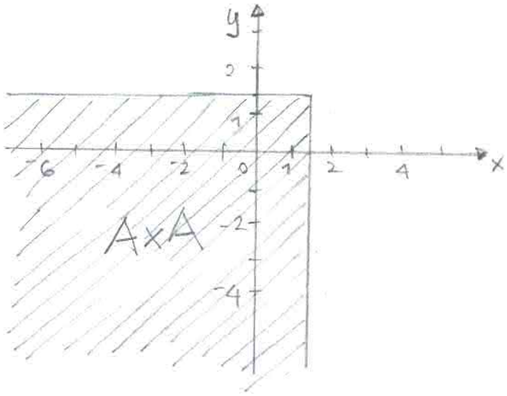
ii) $A = \{x \in \mathbb{R} : f(x) \geq -1\} = \left]-\infty, \frac{3}{2}\right]$ $\qquad B = \{x \in \mathbb{R} : g(x) \leq 0\} = \left]-\infty, -2\right[\cup [-1, 1] \cup \left]2, +\infty\right[$



iii) $A \cap B = \left]-\infty, -2\right[\cup [-1, 1]$ $\qquad A \cup B = \left]-\infty, \frac{3}{2}\right] \cup \left]2, +\infty\right[$

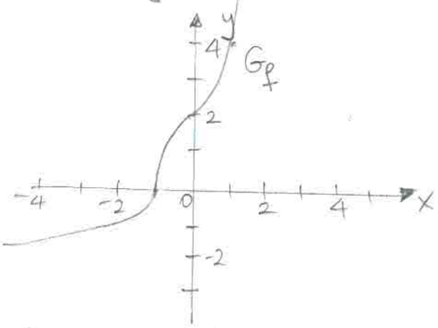
$A \setminus B = \left[-2, -1\right[\cup \left]1, 3/2\right].$

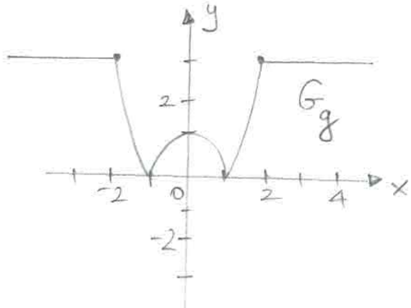
iv)



2) i)

$$f(x)=\begin{cases}\sqrt[3]{x+1} & \text{se } x<-1\\ 2\log_2(x+2) & \text{se } -1\le x\le 0\\ 2^{x+1} & \text{se } x>0;\end{cases}\qquad g(x)=\begin{cases}|x^2-1| & \text{se } x\in[-2,2]\\ 3 & \text{se } x\in\mathbb{R}\setminus[-2,2].\end{cases}$$



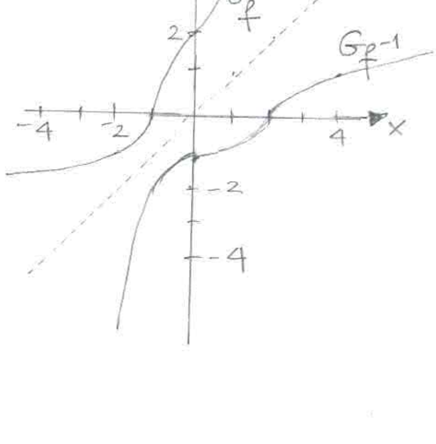


ii) $\forall k\in\mathbb{R}$ $\exists!$ soluzione dell'eq. $f(x)=k$.

iii) $g(\mathbb{R})=\underline{\underline{[0,3]}}$; poiché $g(\mathbb{R})\subsetneq\mathbb{R}$, g non è suriettiva.
Inoltre, g non è iniettiva: infatti, basta prendere, per esempio, $x_1=-1\neq x_2=1$ e $g(x_1)=g(x_2)=0$.

iv) $(g\circ f)(0)\doteq g(f(0))=g(2)=\underline{\underline{3}}$;
$(fg)(1)\doteq f(1)\,g(1)=4\cdot 0=\underline{\underline{0}}$;
$(f+g)(-9)\doteq f(-9)+g(-9)=\sqrt[3]{-8}=-2+3=\underline{\underline{1}}$.

v) $f(\mathbb{R})=\mathbb{R}$, f è iniettiva e suriettiva; $\exists\, f^{-1}:\mathbb{R}\to\mathbb{R}$



vi) $\min\limits_{[-1,2]} f = f(-1)=\underline{\underline{0}}$ $\quad\underline{\underline{x=-1}}$ pt. di minimo

$\max\limits_{[-1,2]} f = f(2)=\underline{\underline{8}}$ $\quad\underline{\underline{x=2}}$ pt. di massimo.

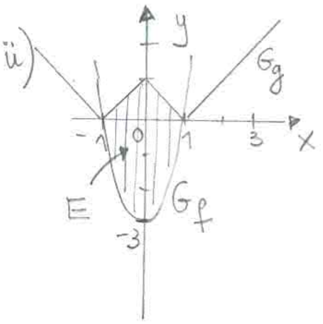
3) i) • $\displaystyle\int_0^1 \left(\frac{e^x}{e^x+3} - x^{1/5}\right) dx = \left[\log(e^x+3) - \frac{x^{6/5}}{6/5}\right]_0^1 =$

$= \log(e+3) - \frac{5}{6} - \log 4 = \log\left(\frac{e+3}{4}\right) - \frac{5}{6}.$

• $\displaystyle\lim_{x\to 0^+} \frac{e^x-1}{\sqrt{x}} = \lim_{x\to 0^+} \underbrace{\frac{e^x-1}{x}}_{\to 1 \text{ limite notevole}} \cdot \sqrt{x} = 0$.

• $\displaystyle\lim_{x\to 0} \frac{e^{3x}-1}{x} = \lim_{x\to 0} \underbrace{\frac{e^{3x}-1}{3x}}_{\to 1} \cdot 3 = 3$. □

ii)



$\displaystyle \text{area}(E) = \int_{-1}^{1} [g(x) - f(x)]\,dx = 2\int_0^1 (g(x)-f(x))\,dx$

$\displaystyle = \cancel{2}\cdot\frac{1\cdot 1}{\cancel{2}} - 2\int_0^1 f(x)\,dx = 1 - 2\int_0^1 (3x^2-3)\,dx$

$= 1 - 2\left[x^3 - 3x\right]_0^1 = 1 - 2(1-3) = 5.$ □

iii) • $\log(3-x^2) + \log_{1/e}(2|x|) < 0 \iff \log(3-x^2) - \log(2|x|) < 0$

$\iff \log(3-x^2) < \log(2|x|) \underset{\log x \uparrow}{\iff} \begin{cases} 3-x^2 > 0 \\ x \neq 0 \\ 3-x^2 < 2|x| \end{cases}$

$\iff \begin{cases} x^2 < 3 \\ x \neq 0 \\ x^2 + 2|x| - 3 > 0 \end{cases} \iff \begin{cases} -\sqrt{3} < x < 0 \\ x^2 - 2x - 3 > 0 \end{cases}$ o $\begin{cases} 0 < x < \sqrt{3} \\ x^2 + 2x - 3 > 0 \end{cases}$

$\iff \begin{cases} -\sqrt{3} < x < 0 \\ x < -1 \text{ o } x > 3 \end{cases}$ o $\begin{cases} 0 < x < \sqrt{3} \\ x < -3 \text{ o } x > 1 \end{cases}$

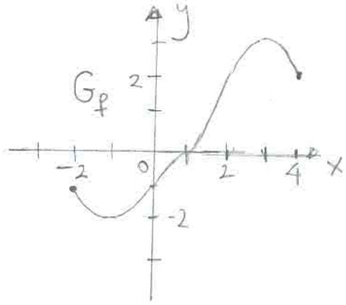
$S = \left]-\sqrt{3}, -1\right[\cup \left]1, \sqrt{3}\right[$. □

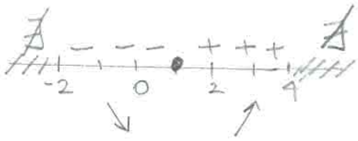
• $\log_2 64 < 6^{|x-3|-1} \iff 6 < 6^{|x-3|-1}$

$\underset{6^x \uparrow}{\iff} 1 < |x-3| - 1$

$\iff |x-3| > 2 \iff x-3 < -2$ o $x-3 > 2$

$\Rightarrow S = \left]-\infty, 1\right[\cup \left]5, +\infty\right[$. □

4) 

i) 

$f = F'$ (teorema fond. del Calcolo Integrale)

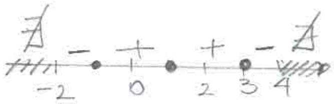
F

Si ha che F ristretta a $[-2,1]$ è decrescente,

F ristretta a $[1,4]$ è crescente. ☐

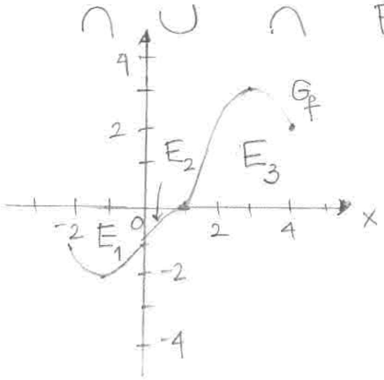
ii) $x=-2$, $x=4$ sono pt. di massimo loc. di F,

$x=1$ è punto di minimo loc. di F. ☐

iii) 

$f' = F''$

$\cap$ $\cup$ $\cap$ F



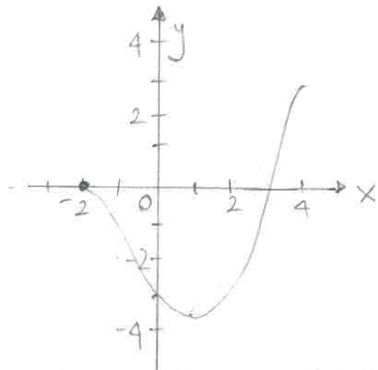
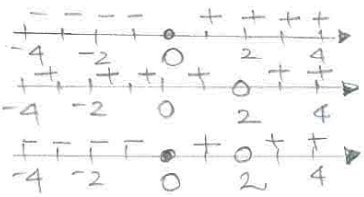


grafico qualitativo di F ☐

iv) $F(0) = -\text{area}(E_1) < -2$. ■

5) i) $f(x) = \dfrac{x}{(x-2)^2}$

- $\text{dom} f = \mathbb{R}\setminus\{2\} =]-\infty, 2[\cup]2, +\infty[$.

- segno f 

- $\lim_{x\to-\infty} f(x) = 0$ $\quad y=0$ asintoto orizzontale per f, per $x\to-\infty$;

  $\lim_{x\to 2^-} f(x) = +\infty$ $\quad x=2$ asintoto verticale per f

  $\lim_{x\to 2^+} f(x) = +\infty$ $\quad x=2$ "

  $\lim_{x\to+\infty} f(x) = 0$ $\quad y=0$ asintoto orizzontale per f, per $x\to+\infty$.

- f è continua in $\mathbb{R}\setminus\{2\}$.

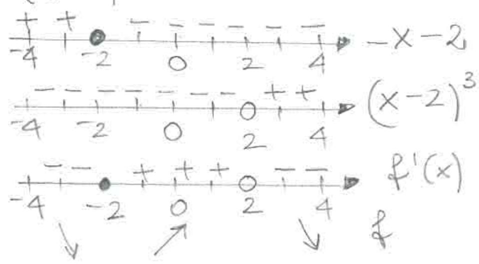
• dom $f' = \mathbb{R}\setminus\{2\}$ $\quad f'(x) = \dfrac{(x-2)^{\cancel{2}} - 2\cancel{(x-2)}\,x}{(x-2)^{\cancel{4}\,3}} =$

$= \dfrac{-x-2}{(x-2)^3}$

$f'(x)=0 \iff \underline{\underline{x=-2}}$ pt. critico per f

$x=-2$ pt. di min. loc. stretto per f

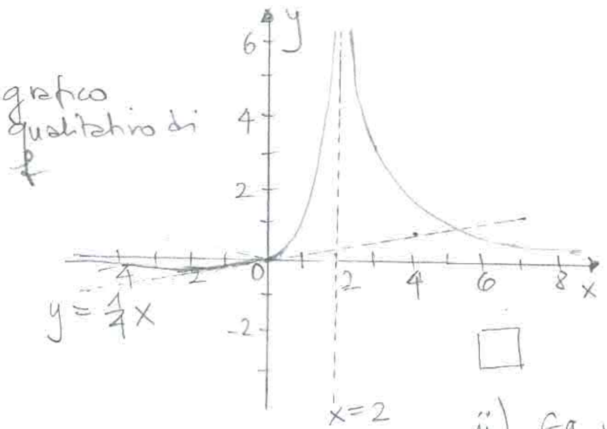
$f(-2) = \dfrac{-2}{16} = \underline{\underline{-\dfrac{1}{8}}}$.

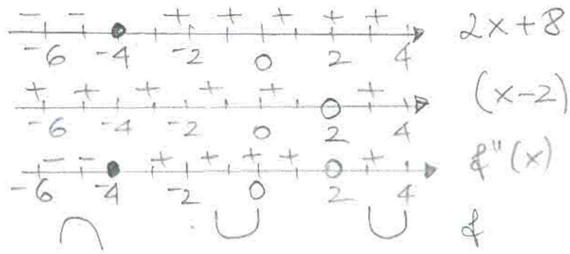


• dom $f'' = \mathbb{R}\setminus\{2\}$ $\quad f''(x) = \dfrac{-(x-2)^{\cancel{3}} - (-x-2)\,3\cancel{(x-2)^2}}{(x-2)^{\cancel{6}\,4}}$

$= \dfrac{2x+8}{(x-2)^4}$

grafico qualitativo di f





ii) Eq. retta tg al grafico di f nel punto $(0,0)$ è $y = f(0) + f'(0)x \Rightarrow \underline{\underline{y = +\frac{1}{4}x}}$.

iii) $\displaystyle\int_{-2}^{-1} \frac{f'(x)}{f(x)}\,dx = \Big[\log|f(x)|\Big]_{-2}^{-1} = \log\frac{1}{9} - \log\frac{1}{8} = \underline{\underline{\log\frac{8}{9}}}$.

6) $\begin{cases} -\dfrac{x^2}{4} + \dfrac{y^2}{4} \le 1 \\ (x+2)^2 + y^2 > 1 \\ x < 2. \end{cases}$