Università degli Studi di Trento – Facoltà di Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
CdL in Interfacce e Tecnologie della Comunicazione – CdL in Filosofia
SECONDA PROVA INTERMEDIA DI ANALISI MAT. (CON EL. DI ALGEBRA
a.a. 2012–2013 — Rovereto, 21 gennaio 2013

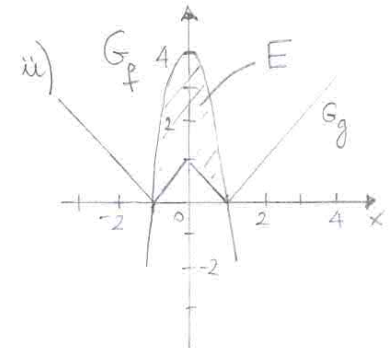
FILA (A)

1) i) $\displaystyle\int_1^2 \left(\frac{e^x}{e^x+3} + x^{-5}\right) dx = \left[\log(e^x+3) + \frac{x^{-4}}{-4}\right]_1^2 = \left(\log(e^2+3) - \frac{1}{4\cdot 2^4}\right) -$

$$- \left(\log(e+3) - \frac{1}{4}\right) = \log\left(\frac{e^3+3}{e+3}\right) + \frac{1}{4} - \frac{1}{64} = \log\left(\frac{e^3+3}{e+3}\right) + \frac{15}{64}.$$

$$\lim_{x\to 0^+} \frac{e^x-1}{x^2} = \lim_{x\to 0^+} \underbrace{\left(\frac{e^x-1}{x}\right)}_{\to 1 \text{ (limite notevole)}} \cdot \frac{1}{x} = +\infty$$

$$\lim_{x\to 0} \frac{e^{3x}-1}{4x} = \lim_{x\to 0} \underbrace{\left(\frac{e^{3x}-1}{3x}\right)}_{\to 1} \cdot \frac{3}{4} = \frac{3}{4}.$$

ii) 

$$\text{area}(E) = \int_{-1}^{1} (f(x)-g(x))\,dx = 2\int_0^1 (f(x)-g(x))\,dx$$

$$= 2\int_0^1 f(x)\,dx - \frac{1\cdot 1}{\cancel{2}}\cdot\cancel{2}$$

$$= 2\int_0^1 (-4x^2+4)\,dx - 1 =$$

$$= 2\left[-4\frac{x^3}{3} + 4x\right]_0^1 - 1 = 2\left(-\frac{4}{3}+4\right) - 1 = \frac{16}{3} - 1 = \frac{13}{3}$$

iii) $\displaystyle\sum_{n=1}^{5} \int_0^1 (e^{nx} + x^n)\,dx = \sum_{n=1}^{5} \left[\frac{e^{nx}}{n} + \frac{x^{n+1}}{n+1}\right]_0^1 = \sum_{n=1}^{5} \left[\left(\frac{e^n}{n} + \frac{1}{n+1}\right) - \left(\frac{1}{n}\right)\right]$

$$= \left(e + \cancel{\tfrac{1}{2}} - 1\right) + \left(\frac{e^2}{2} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{2}}\right) + \left(\frac{e^3}{3} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{3}}\right) + \left(\frac{e^4}{4} + \cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{4}}\right) +$$

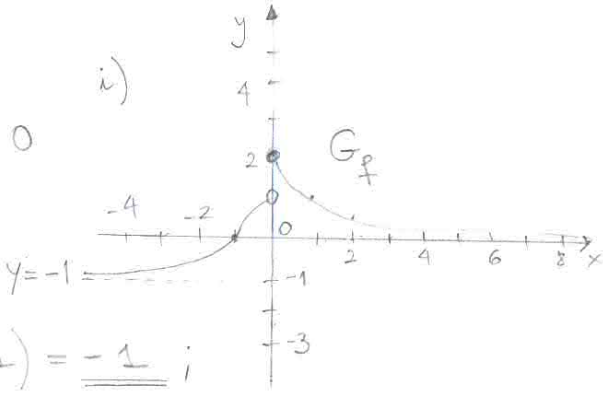
$$+ \left(\frac{e^5}{5} + \frac{1}{6} - \cancel{\tfrac{1}{5}}\right) = e + \frac{e^2}{2} + \frac{e^3}{3} + \frac{e^4}{4} + \frac{e^5}{5} - \frac{5}{6}.$$

iv) $\displaystyle\frac{x^3}{2!} - \frac{x^6}{3!} + \frac{x^9}{4!} - \cdots + \frac{x^{27}}{10!} = \sum_{n=1}^{9} (-1)^{n+1} \frac{x^{3n}}{(n+1)!}$

2) $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases}\frac{1}{x^2}-1 & \text{se } x\le -1\\ \log_2(x+2) & \text{se } -1<x<0\\ 2^{-x+1} & \text{se } x\ge 0\end{cases}$$

i)



ii) $\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\left(\frac{1}{x^2}-1\right)=\underline{\underline{-1}}$;

$\lim_{x\to-1^+} f(x)=\lim_{x\to-1^+}\left(\log_2(x+2)\right)=\log_2(-1+2)=\underline{\underline{0}}$;

$\lim_{x\to0^-} f(x)=\lim_{x\to0^-}\left(\log_2(x+2)\right)=\log_2 2=\underline{\underline{1}}$;

$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty} 2^{-x+1}=\underline{\underline{0}}$.

iii) $\min_{[-2,2]} f=f(-2)=\frac{1}{4}-1=\underline{\underline{-\frac{3}{4}}}$ $\qquad x=-2$ pt. di minimo;

$\max_{[-2,2]} f=f(0)=\underline{\underline{2}}$ $\qquad x=0$ pt. di massimo.

f non verifica le ipotesi del teorema di Weierstrass su $[-2,2]$, non essendo continua in $x=0$; infatti $\lim_{x\to0^-} f(x)=1\neq\lim_{x\to0^+} f(x)=$ $=2=f(0)$.

iv) $\int_1^2 f(x)\,dx=\int_1^2 2^{-x+1}\,dx=2\int_1^2 2^{-x}\,dx=\left[-2\cdot\frac{2^{-x}}{\log 2}\right]_1^2=$

$=\frac{-\cancel{2}^{1}}{\cancel{4}_{2}\log 2}+\frac{\cancel{2}^{1}}{\cancel{2}_{1}\log 2}=\underline{\underline{\frac{1}{2\log 2}}}$.

3) • $(|x-2|-x^2)(|x|-1)\ge 0$

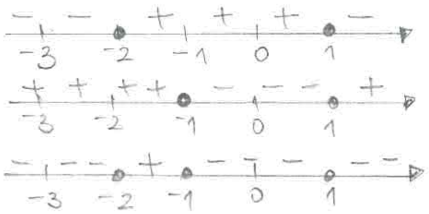
$|x-2|-x^2\ge 0 \iff \begin{cases}x\ge 2\\ x-2-x^2\ge 0\end{cases}$ o $\begin{cases}x<2\\ -x+2-x^2\ge 0\end{cases}$

$\iff \begin{cases}x\ge 2\\ x^2-x+2\le 0\end{cases}$ o $\begin{cases}x<2\\ x^2+x-2\le 0\end{cases}$

$\iff \begin{cases}x\ge 2\\ \emptyset\end{cases}$ o $\begin{cases}x<2\\ -2\le x\le 1\end{cases}$

quindi $|x-2|-x^2\ge 0 \iff -2\le x\le 1$

D'altra parte $|x|-1 \geq 0 \iff |x| \geq 1 \iff x \leq -1$ o $x \geq 1$



$\underline{\underline{S = [-2,-1] \cup \{1\}.}}$

- $\log(2-x^2) + \log_{1/e}(|x|) > 0 \iff \log(2-x^2) - \log(|x|) > 0$

$$\iff \log(2-x^2) > \log(|x|) \underset{\log x \nearrow}{\iff} \begin{cases} 2-x^2 > 0 \\ x \neq 0 \\ 2-x^2 > |x| \end{cases}$$

$$\iff \begin{cases} x^2 < 2 \\ x \neq 0 \\ x^2 + |x| - 2 < 0 \end{cases} \iff \begin{cases} -\sqrt{2} < x < 0 \\ x^2 - x - 2 < 0 \end{cases} \text{ o } \begin{cases} 0 < x < \sqrt{2} \\ x^2 + x - 2 < 0 \end{cases}$$

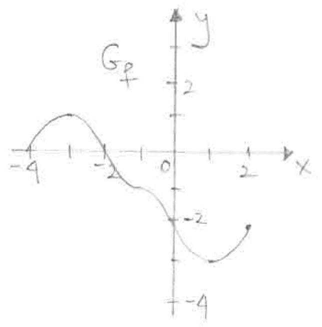
$$\iff \begin{cases} -\sqrt{2} < x < 0 \\ -1 < x < 2 \end{cases} \text{ o } \begin{cases} 0 < x < \sqrt{2} \\ -2 < x < 1 \end{cases} \implies S =]-1,0[\cup]0,1[$$

$$= \underline{\underline{]-1,1[\setminus \{0\}.}}$$

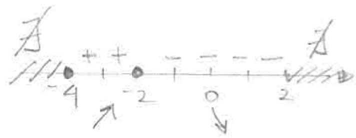
- $\log_2 32 = 5^{x^2-1} \iff 5 = 5^{x^2-1} \iff 1 = x^2-1$

$$\iff x^2 = 2 \qquad \underline{\underline{S = \{-\sqrt{2}, \sqrt{2}\}.}}$$

4)



i)



$f = F'$ (teorema fond. del calcolo integral)

F

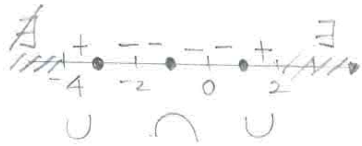
Si ha che F ristretta a $[-4,-2]$ è crescente,

F ristretta a $[-2,2]$ è decrescente.

ii) $x=-4$, $x=2$ sono pt. di minimo loc. di F,

$x=-2$ è punto di massimo loc. di F.

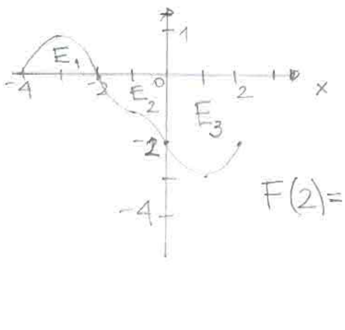
iii)



$f' = F''$

F

iv)



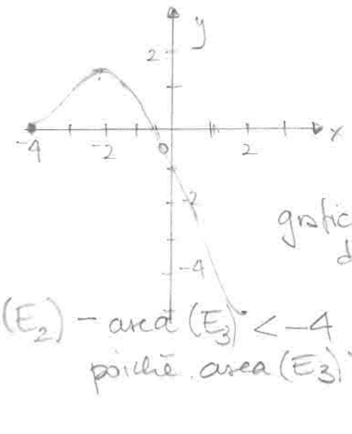


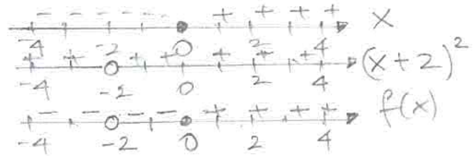
grafico qualitativo di F

$$F(2) = \underbrace{\text{area}(E_1) - \text{area}(E_2)}_{<0} - \text{area}(E_3) < -4$$

poiché $\text{area}(E_3) > 2 \cdot 2 = 4$

5) i) $f(x) = \dfrac{x}{(x+2)^2}$ • dom $f = \mathbb{R}\setminus\{-2\} =]-\infty,-2[\,\cup\,]-2,+\infty[$.

• segno f



• $\lim_{x\to-\infty} f(x) = 0$ $y=0$ asintoto orizzontale per f, per $x\to-\infty$

$\lim_{x\to-2^-} f(x) = -\infty$ $x=-2$ asintoto verticale per f

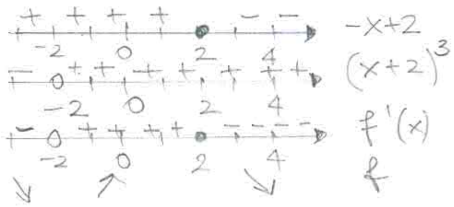
$\lim_{x\to-2^+} f(x) = -\infty$ "

$\lim_{x\to+\infty} f(x) = 0$ $y=0$ asintoto orizzontale per f, per $x\to+\infty$.

• f è continua su $\mathbb{R}\setminus\{-2\}$.

• dom $f' = \mathbb{R}\setminus\{-2\}$ $f'(x) = \dfrac{(x+2)^{\cancel{2}} - x\cdot 2\cancel{(x+2)}}{(x+2)^{\cancel{4}\,3}} = \dfrac{-x+2}{(x+2)^3}$

$f'(x)=0 \iff \underline{\underline{x=2 \text{ pt. critico per } f}}$

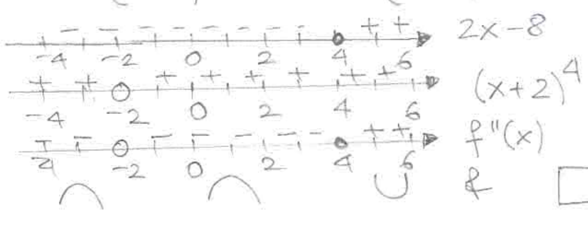


$x=2$ pt. di massimo loc. stretto per f

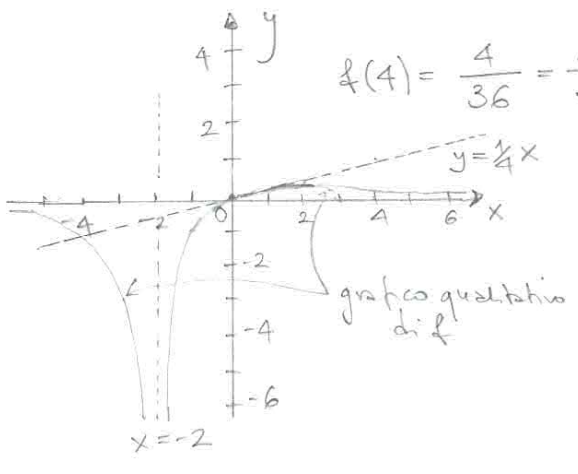
$f(2) = \dfrac{2}{4^2} = \underline{\underline{\dfrac{1}{8}}}$

• dom $f'' = \mathbb{R}\setminus\{-2\}$ $f''(x) = \dfrac{-1(x+2)^{\cancel{3}} - (-x+2)\,3\cancel{(x+2)^2}}{(x+2)^{\cancel{6}\,4}} =$

$= \dfrac{-x-2+3x-6}{(x+2)^4} = \dfrac{2x-8}{(x+2)^4}$



$f(4) = \dfrac{4}{36} = \dfrac{1}{9}$



ii) Eq. retta tg ad f nel pt. $(0,0)$

$y = f(0) + f'(0)x \Rightarrow y = \frac{1}{4}x$

iii) $\displaystyle\int_1^2 \frac{f'(x)}{f(x)}\,dx = \Big[\log|f(x)|\Big]_1^2 = \log\left(\frac{1}{8}\right) - \log\left(\frac{1}{9}\right)$

$= \underline{\underline{\log\frac{9}{8}}}$.

6) $C_{8,3} = \dfrac{8!}{3!\,5!} = \dfrac{8\cdot7\cdot\cancel{6}\cdot\cancel{5!}}{\cancel{6}\cdot\cancel{5!}} = \underline{\underline{56}}$.