Università di Trento — Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
ESAME SCRITTO DI ANALISI MATEMATICA
a.a. 2013-2014 — Rovereto, 23 gennaio 2014

FILA (B)

1) i) $\mathcal{A}$ = "$\exists x, y \in \mathbb{R} : [(x > y) \text{ e } (|x| < |y|)]$"

non $\mathcal{A}$ = "$\forall x, y \in \mathbb{R}$, non$(x > y)$ o non$(|x| < |y|)$"

= "$\forall x, y \in \mathbb{R}$, $(x \leq y)$ o $(|x| \geq |y|)$".

La proposizione $\mathcal{A}$ è vera: basta prendere $x = -2$, $y = -3$ e si ha $x > y$ e $|x| < |y|$. □

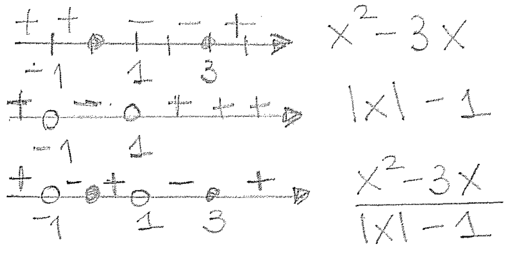
ii) a) $A = \{x \in \mathbb{R} : \sqrt{x - 2x^2} \leq 1\} = \{x \in \mathbb{R} : 0 \leq x - 2x^2 \leq 1\}$

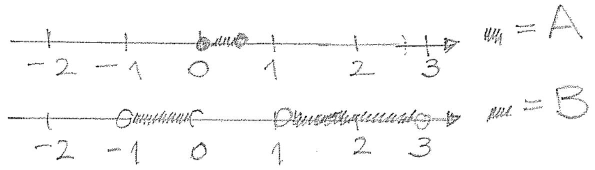
$= \{x \in \mathbb{R} : 0 \geq 2x^2 - x \geq -1\}$

$$\begin{cases} 2x^2 - x \leq 0 \\ 2x^2 - x + 1 \geq 0 \end{cases} \iff \begin{cases} x \in [0, \frac{1}{2}] \\ x \in \mathbb{R} \end{cases} \Rightarrow \underline{A = [0, \tfrac{1}{2}]}.$$

$\underline{B} = \{x \in \mathbb{R} : \dfrac{x^2 - 3x}{|x| - 1} < 0\}$

$= \underline{]-1, 0[\cup]1, 3[}.$

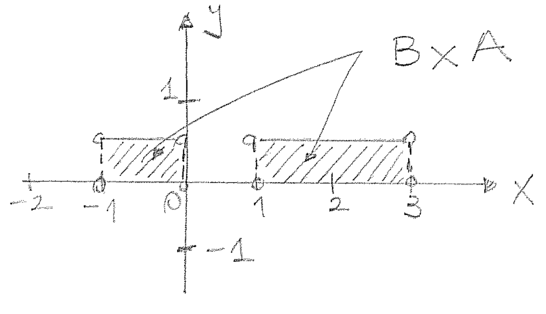




A e B sono insiemi limitati. □

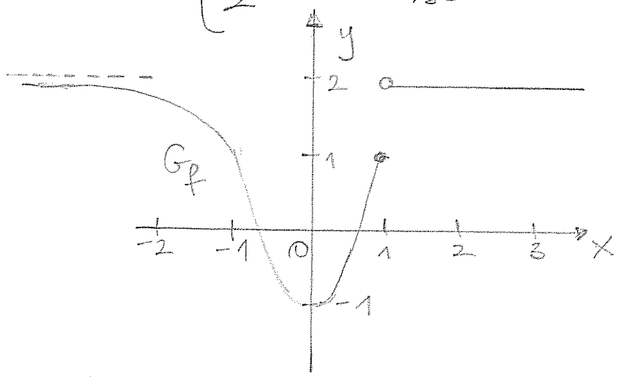
b) $\underline{A \cup B =]-1, \frac{1}{2}] \cup]1, 3[}$; $A \cup B$ non è un intervallo. □

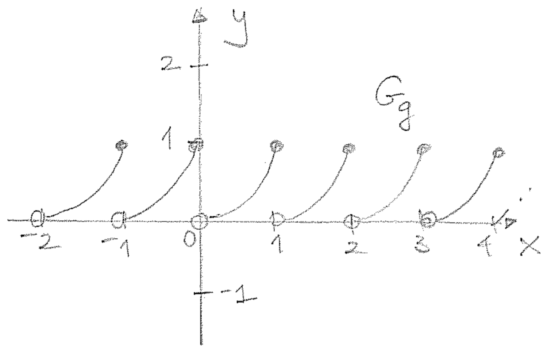
c)



2) $f: \mathbb{R} \to \mathbb{R}$ $\qquad$ $g: \mathbb{R} \to \mathbb{R}$

i)
$$f(x) = \begin{cases} \frac{1}{x} + 2 & \text{se } x < -1 \\ 2x^4 - 1 & \text{se } -1 \le x \le 1 \\ 2 & \text{se } x > 1 \end{cases} \qquad g(x) = (x-k)^2 \quad \text{se } k < x \le k+1,\ k \in \mathbb{Z}$$





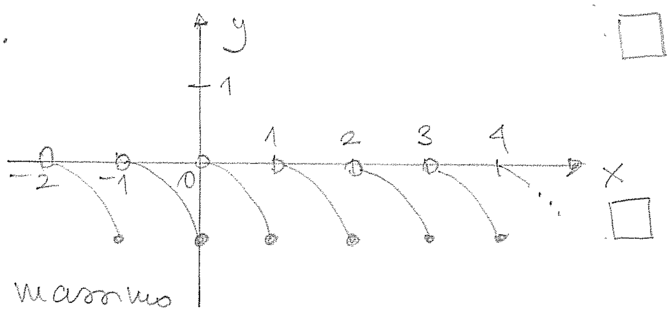
f e g sono entrambe limitate: $-1 \le f(x) \le 2 \quad \forall x \in \mathbb{R}$

$0 \le g(x) \le 1 \quad \forall x \in \mathbb{R}$. □

ii) $f(\mathbb{R}) = \underline{[-1, 2]} \subsetneq \mathbb{R}$; f non è suriettiva.

f non è iniettiva; basta prendere $x_1 = 2$; $x_2 = 3$ e si ha $f(x_1) = f(x_2) = 2$. □

iii) $x \longmapsto -g(x+1)$
$\quad \cap\, \mathbb{R}$



□

iv) $\max_{\mathbb{R}} g = 1$ $\qquad x = k \in \mathbb{Z}$
sono i pti di massimo

$\nexists \min_{\mathbb{R}} g$. ■

3) i) • $\log_3 (x^2 - 9) + \log_{1/3} (x+3) \le 2 \iff$
$\log_3 (x^2 - 9) - \log_3 (x+3) \le \log_3 9$

$$\iff \begin{cases} x^2 - 9 > 0 \\ x + 3 > 0 \\ \frac{x^2 - 9}{x+3} \le 9 \end{cases} \iff \begin{cases} x^2 > 9 \\ x > -3 \\ \frac{(x-3)\cancel{(x+3)}}{\cancel{x+3}} \le 9 \end{cases} \iff \begin{cases} x < -3 \text{ o } x > 3 \\ x > -3 \\ x \le 12 \end{cases}$$

$\Rightarrow \underline{S =]3, 12]}$. □

- $4^{x^2} \cdot 2^{|x^2-1|} > 2^{\log_2 4} \iff 2^{2x^2+|x^2-1|} > 2^2$

$\iff 2x^2+|x^2-1| > 2$

$\iff 2x^2+|x^2-1|-2>0$

$\iff \begin{cases} x^2-1\geq 0 \\ 3x^2>3 \end{cases} \quad \text{o} \quad \begin{cases} x^2-1<0 \\ x^2>1 \end{cases}$

$\iff \begin{cases} x^2\geq 1 \\ x^2>1 \end{cases} \quad \text{o} \quad \{\ \emptyset$

$\underline{S = \,]-\infty,-1[\,\cup\,]1,+\infty[}$

$\square$

ii) $\sum_{n=1}^{5} f\left((-1)^n \frac{1}{n}\right)$, dove $f(x)=\begin{cases} 2-|x| & \text{se } x\leq 0 \\ \log x & \text{se } x>0 \end{cases}$

$\sum_{n=1}^{5} f\left((-1)^n \frac{1}{n}\right) = f(-1)+f\left(\frac{1}{2}\right)+f\left(-\frac{1}{3}\right)+f\left(\frac{1}{4}\right)+f\left(-\frac{1}{5}\right)$

$= (2-1)+\log\frac{1}{2}+\left(2-\frac{1}{3}\right)+\log\frac{1}{4}+\left(2-\frac{1}{5}\right)=$

$= 5-\frac{8}{15}-\log 8 = \underline{\frac{67}{15}}-\log 8.$ $\square$

4) i) $f:[-1,1]\to\mathbb{R}$

$f(x)=\begin{cases} -x^2-bx+c & \text{se } -1\leq x<0 \\ d & \text{se } x=0 \\ 3e^x & \text{se } 0<x\leq 1 \end{cases}$

è continua in $x=0 \iff$

$\lim_{x\to 0^-}(-x^2-bx+c)=\lim_{x\to 0^+}3e^x$

$=f(0)=d$

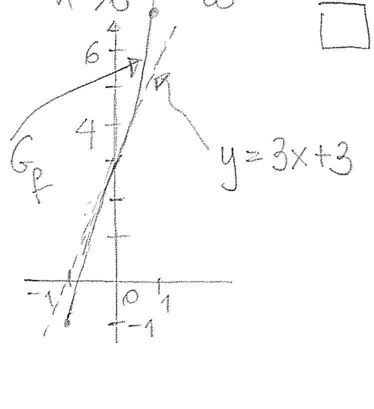
$\iff \underline{c=3=d}.$

$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{-h^2-bh+\cancel{3}-\cancel{3}}{h}=-b$ $\boxed{b=-3}$

$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{3e^h-3}{h}=3\lim_{h\to 0^+}\frac{e^h-1}{h}=3$

$\square$

ii) $f(x)=\begin{cases} -x^2+3x+3 & \text{se } -1\leq x\leq 0 \\ 3e^x & \text{se } 0<x\leq 1 \end{cases}$



iii) $f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=3$ (vedi pt. i));

$y = 3 + 3(x-0)$ è l'eq. della retta tg al grafico di f nel pt. di coordinata $x=0$, ossia $\underline{y=3x+3}$. □

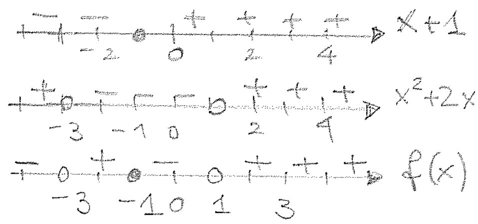
iv) $\min\limits_{[-1,1]} f = \underline{\underline{-1}}$ $\quad \underline{\underline{x=-1}}$ pt. di minimo di f su $[-1,1]$;

$\max\limits_{[-1,1]} f = \underline{\underline{3e}}$ $\quad \underline{\underline{x=1}}$ pt. di massimo di f su $[-1,1]$. □

v) $$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0}(-x^2+3x+3)\,dx + \int_{0}^{1} 3e^x\,dx = \left[-\frac{x^3}{3}+3\frac{x^2}{2}+3x\right]_{-1}^{0}$$
$$+\left[3e^x\right]_0^1 = -\left(\frac{1}{3}+\frac{3}{2}-3\right)+\left(3e-3\right) = \underline{\underline{3e-\frac{11}{6}}}.$$ ■

5) i) $f(x) = \dfrac{x+1}{x^2+2x-3} = \dfrac{x+1}{(x+3)(x-1)}$ $\quad$ • dom $f = \underline{\mathbb{R}\setminus\{-3,1\}}$

• segno di f



• $\lim\limits_{x\to-\infty\,(+\infty)} f(x) = 0$ $\quad y=0$ asintoto orizz. per f, per $x\to-\infty$ $(+\infty)$;

$\lim\limits_{x\to-3^-} f(x) = -\infty$ $\quad x=-3$ asintoto verticale per f;

$\lim\limits_{x\to-3^+} f(x) = +\infty$ $\quad$ "

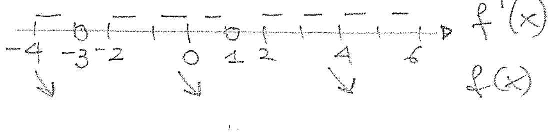
$\lim\limits_{x\to1^-} f(x) = -\infty$ $\quad x=1$ asintoto verticale per f;

$\lim\limits_{x\to1^+} f(x) = +\infty$ $\quad x=1$ asintoto verticale per f.

• f è continua su $\mathbb{R}\setminus\{-3,1\}$.

• f è derivabile su $\mathbb{R}\setminus\{-3,1\}$ e $f'(x) = \dfrac{x^2+2x-3-(x+1)(2x+2)}{(x^2+2x-3)^2} =$

$$= \frac{x^2+2x-3-2x^2-4x-2}{(x^2+2x-3)^2} = \frac{-x^2-2x-5}{(x^2+2x-3)^2}.$$

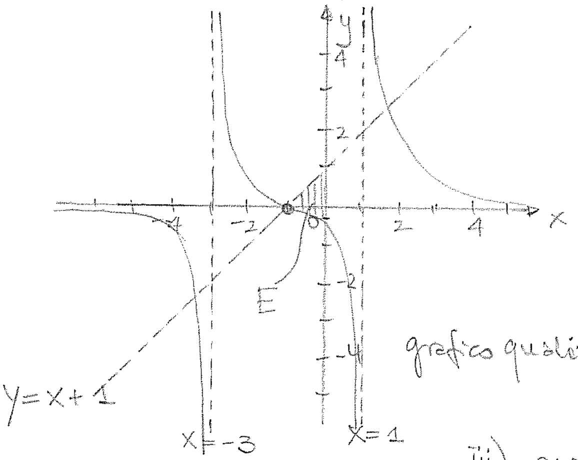


• dom $f'' = \mathbb{R}\setminus\{-3,1\}$

$$f''(x) = \frac{(-2x-2)(x^2+2x-3)^{\cancel{2}} - (-x^2-2x-5)\,2\,\cancel{(x^2+2x-3)}\,(2x+2)}{(x^2+2x-3)^{\cancel{4}3}}$$

$$= \frac{-2x^3-\cancel{4x^2}+6x-2x^2-4x+6+4x^3+\cancel{4x^2}+8x^2+8x+20x+20}{(x^2+2x-3)^3}$$

$$= \frac{2x^3+6x^2+30x+26}{(x^2+2x-3)^3} = \frac{(x+1)\overbrace{(2x^2+4x+26)}^{>0}}{(x^2+2x-3)^3}$$



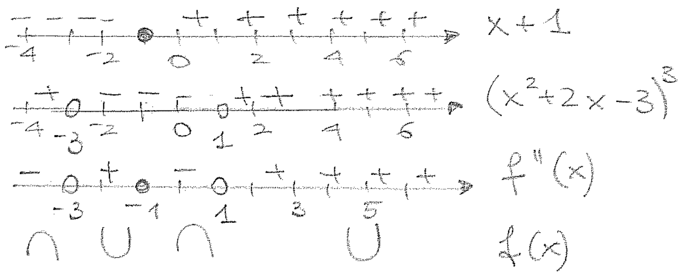


grafico qualitativo di f

iii) $\text{area}(E) = \int_{-1}^{0} (x+1-f(x))\,dx = \left[\frac{x^2}{2}+x-\frac{1}{2}\log|x^2+2x-3|\right]_{-1}^{0} =$

$$= \left(-\frac{1}{2}\log 3\right) - \left(\frac{1}{2}-1-\frac{1}{2}\log 4\right) = \underline{\frac{1}{2}\left(\log\frac{4}{3}+1\right)}.$$

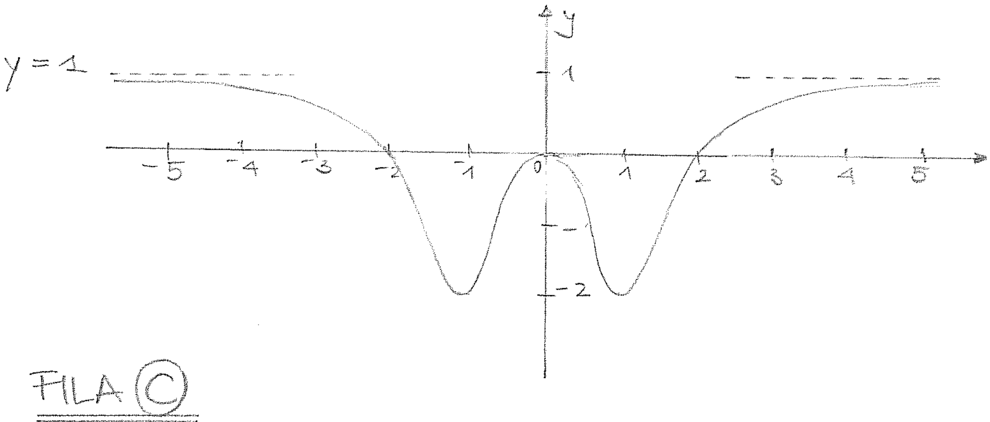
6) $f:\mathbb{R}\to\mathbb{R}$, continua e derivabile:

f è pari, ossia $f(x)=f(-x)\ \forall x\in\mathbb{R}$

$f(1)=-2$ (e quindi $f(-1)=-2$)

$f'(1)=0$ (" $f'(-1)=0$)

$\lim_{x\to+\infty} f(x)=1$ (" $\lim_{x\to-\infty} f(x)=1$)



una possibile funzione f

FILA (C)

1) i) $\mathcal{A}$ = "$\exists x,y\in\mathbb{R}: [(x>y) \text{ e } (x^2<y^2)]$".

non $\mathcal{A}$ = "$\forall x,y\in\mathbb{R}$, non $(x>y)$ o non $(x^2<y^2)$"

= "$\forall x,y\in\mathbb{R}$, $(x\le y)$ o $(x^2\ge y^2)$".

La proposizione $\mathcal{A}$ è vera: basta prendere $x=-2$, $y=-3$ e si ha $x>y$ e $x^2<y^2$.

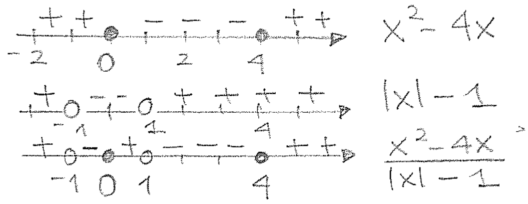
ii) a) $A = \{x \in \mathbb{R} : \sqrt{x - x^2} \leq 1\} = \{x \in \mathbb{R} : 0 \leq x - x^2 \leq 1\}$

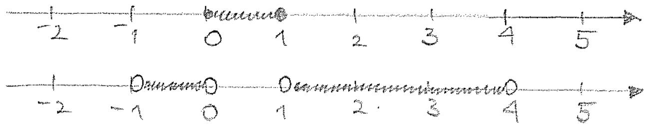
$= \{x \in \mathbb{R} : 0 \geq x^2 - x \geq -1\}$

$$\begin{cases} x^2 - x \leq 0 \\ x^2 - x + 1 \geq 0 \end{cases} \iff \begin{cases} x \in [0,1] \\ x \in \mathbb{R} \end{cases} \Rightarrow \underline{A = [0,1]}.$$

$\underline{B} = \left\{x \in \mathbb{R} : \dfrac{x^2 - 4x}{|x| - 1} < 0\right\}$

$= \underline{\,]-1, 0[\, \cup \,]1, 4[\,}.$

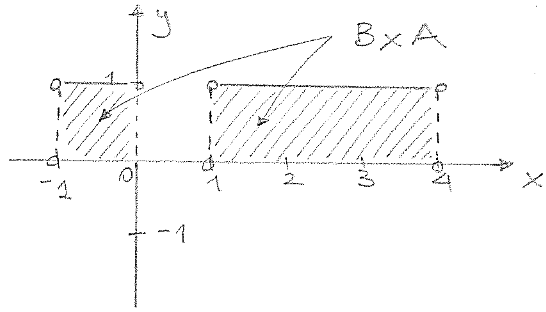




A e B sono insiemi limitati.

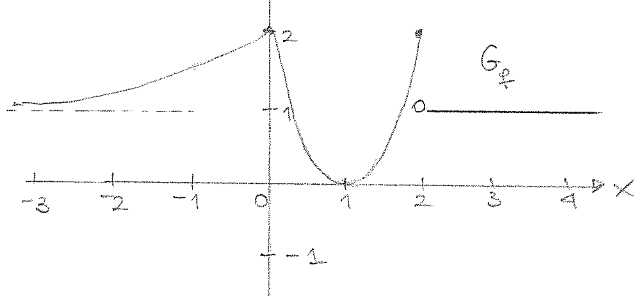
b) $\underline{A \cup B = \,]-1, 4[}$; $A \cup B$ è un intervallo.

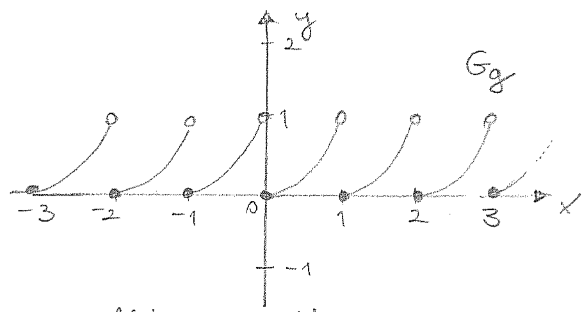
c)



2) $f : \mathbb{R} \to \mathbb{R}$ $\qquad$ $g : \mathbb{R} \to \mathbb{R}$

i)
$$f(x) = \begin{cases} 2^x + 1 & \text{se } x < 0 \\ 2(x-1)^4 & \text{se } 0 \leq x \leq 2 \\ 1 & \text{se } x > 2 \end{cases} \qquad g(x) = (x-k)^2 \quad \text{se } k \leq x < k+1, \; k \in \mathbb{Z}$$



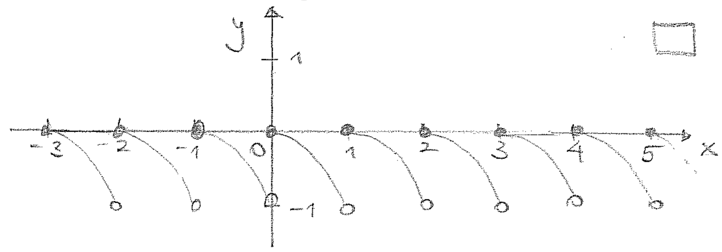


f e g sono entrambe limitate: $0 \leq f(x) \leq 2 \quad \forall x \in \mathbb{R}$

$0 \leq g(x) < 1 \quad \forall x \in \mathbb{R}$.

ii) $f(\mathbb{R}) = [0, 2] \subsetneq \mathbb{R}$; f non è suriettiva.

f non è iniettiva; basta prendere $x_1 = 3$, $x_2 = 4$ e risulta $f(x_1) = f(x_2) = 2$.



iii) $\underset{\mathbb{R} \ni}{x} \longmapsto -g(x+1)$

iv) $\max_{\mathbb{R}} g \;\nexists$

$\min_{\mathbb{R}} g = 0$ $x = k \in \mathbb{Z}$ sono pt. di minimo.

3) i) vedi SPI, FILA (A) Es. 3):

$\log_2(x^2-4) + \log_{\frac{1}{2}}(x+2) \leq 3$ $\underline{\underline{S =]2, 10]}}$.

$9^{x^2} \cdot 3^{|x^2-1|} < 3^{\log_3 9}$ $\underline{\underline{S =]-1, 1[}}$.

ii) $\sum_{n=1}^{5} f\left((-1)^n \frac{1}{n}\right)$, dove $f(x) = \begin{cases} 1 - |x| & \text{se } x \leq 0 \\ \log x & \text{se } x > 0 \end{cases}$.

$$\sum_{n=1}^{5} f\left((-1)^n \frac{1}{n}\right) = f(-1) + f\left(\frac{1}{2}\right) + f\left(-\frac{1}{3}\right) + f\left(\frac{1}{4}\right) + f\left(-\frac{1}{5}\right) =$$

$$= \cancel{(1-1)} + \log\frac{1}{2} + \left(1 - \frac{1}{3}\right) + \log\frac{1}{4} + \left(1 - \frac{1}{5}\right) = \frac{22}{15} - \log 8.$$

4) i) $f : [-1, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} x^2 + bx - c & \text{se } -1 \leq x < 0 \\ d & \text{se } x = 0 \\ -2e^x & \text{se } 0 < x \leq 1 \end{cases}$$

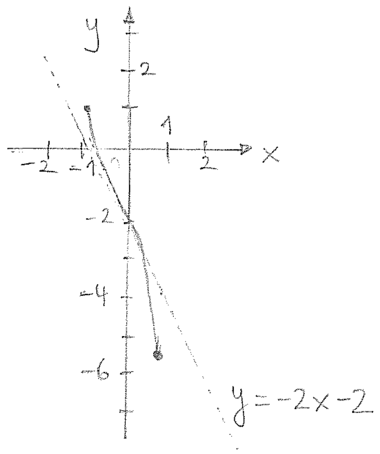
è continua in $x = 0 \iff \lim_{x \to 0^-} (x^2 + bx - c) = \lim_{x \to 0^+} (-2e^x) = f(0) = d$

$\iff \underline{\underline{-c = -2 = d}}$

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 + bh \cancel{-2} \cancel{+2}}{h} = b \qquad \boxed{b = -2}$$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{-2e^h + 2}{h} = \lim_{h \to 0^+} \frac{-2(e^h - 1)}{h} = -2$$

ii) $f(x) = \begin{cases} x^2 - 2x - 2 & \text{se } -1 \leq x \leq 0 \\ -2e^x & \text{se } 0 < x \leq 1 \end{cases}$



iii) $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = -2$

(vedi pt. i)) ;

$y = -2 - 2(x - 0)$ è l'eq. della retta tg. al grafico di f nel pt. di coordinata $x = 0$, ossia $y = -2x - 2$. □

iv) $\min_{[-1,1]} f = -2e$ $\quad x = 1$ pt. di minimo di f su $[-1,1]$.

$\max_{[-1,1]} f = 1$ $\quad x = -1$ pt. di massimo di f su $[-1,1]$. □

v) $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (x^2 - 2x - 2)\,dx + \int_{0}^{1} (-2e^x)\,dx = \left[\frac{x^3}{3} - x^2 - 2x\right]_{-1}^{0} +$

$+ \left[-2e^x\right]_0^1 = -\left(-\frac{1}{3} - 1 + 2\right) + \left(-2e + 2\right) = -2e + \frac{4}{3}$

5) vedi SPI, FILA A, Es. 5).

6) $f : \mathbb{R} \to \mathbb{R}$ continua e derivabile :

f è pari, ossia $f(x) = f(-x) \quad \forall x \in \mathbb{R}$

$f(3) = 1$ $\qquad$ (e quindi $f(-3) = 1$)

$f'(3) = 0$ $\qquad$ (" $f'(3) = 0$)

$\lim_{x \to +\infty} f(x) = -2$ $\qquad$ ($\lim_{x \to -\infty} f(x) = -2$)