Università di Trento - Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
ESAME SCRITTO DI ANALISI MATEMATICA
a.a. 2013-2014 - Rovereto, 5 giugno 2014

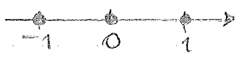
FILA (A)

1) i) $\mathcal{A}$ = " $\exists x_1, x_2 \in \mathbb{R}$: $[x_1 \neq x_2$ e $|x_1 - 1| = |x_2 + 1|]$ ".

non $\mathcal{A}$ = " $\forall x_1, x_2 \in \mathbb{R}$, $x_1 = x_2$ o $|x_1 - 1| \neq |x_2 + 1|$ ".

Osserviamo che $\mathcal{A}$ è la proposizione vera : infatti, basta prendere $x_1 = 1$ e $x_2 = -1$ (ma qualsiasi altra coppia $x_1 = x$, $x_2 = -x$ con $x \in \mathbb{R}$ va bene). $\square$

ii) $A = \{-1, 0, 1\}$



A = B

$B = \{x \in \mathbb{Z} : x^2 \leq 1\} = \{-1, 0, 1\} = A$

$C = [-1, 1]$

C

$[0,1] \subset A$ (F) : $]0,1[\not\subset A$.

$(-1,-1) \in B \times A$ (V) poiché $-1 \in A$, $-1 \in B$.

$\{0,1\} \in \mathcal{P}(C)$ (V) poiché $0 \in C$ e $1 \in C$.

$]0,1[\in \mathcal{P}(B)$ (F) poiché $]0,1[\not\subset B$.

$A \times B \subseteq C \times C$ (V) poiché $A \subset C$, $B \subset C$.

$A = B$ (V)

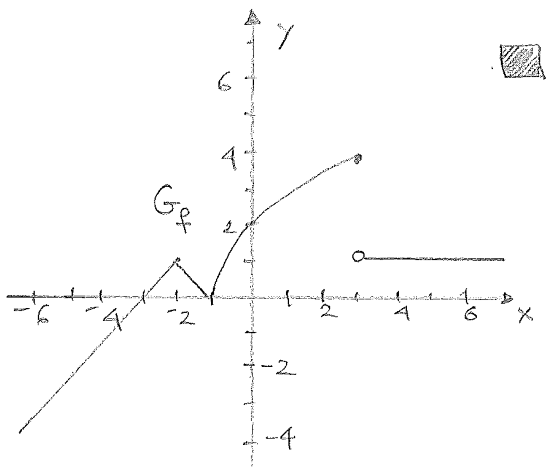
$A \subset B$ (V)

$-2 \in B$ (F) . $\blacksquare$

2) $f : \mathbb{R} \to \mathbb{R}$

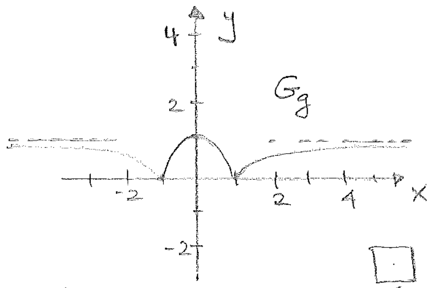
i)

$$f(x) = \begin{cases} -|x+2| + 1 & \text{se } x < -1 \\ 2\sqrt{x+1} & \text{se } -1 \leq x \leq 3 \\ 1 & \text{se } x > 3 \end{cases}$$



$g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} -x^2+1 & \text{se } -1 \le x \le 1 \\ -\dfrac{1}{x^2}+1 & \text{se } x<-1 \text{ o } x>1 \end{cases}$$



ii) $\underline{f(\mathbb{R}) =]-\infty, 4]}$. f è limitata superiormente, ma non è limitata inferiormente; quindi $\underline{f \text{ non è limitata}}$.

iii) $(g\circ f)(-2) \underset{\uparrow}{=} g(f(-2)) = g(1) = \underline{\underline{0}}$;

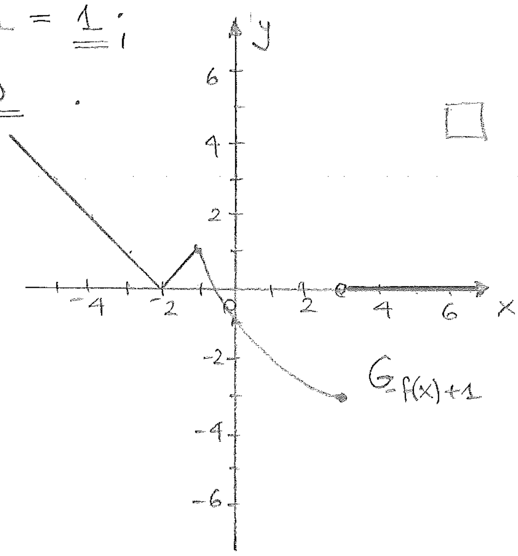
dom $g = \mathbb{R} \supset f(\mathbb{R})$

$(f-g)(0) = f(0) - g(0) = 2 - 1 = \underline{\underline{1}}$;

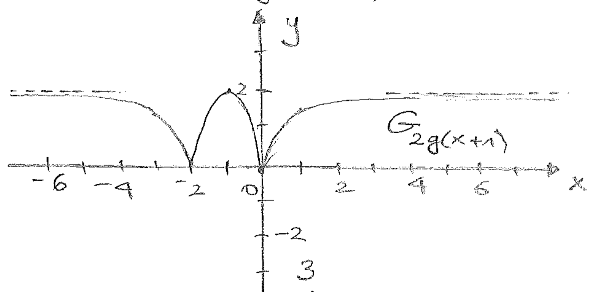
$\left(\dfrac{f}{g}\right)(-3) \underset{\nearrow}{=} \dfrac{f(-3)}{g(-3)} = \dfrac{0}{g(-3)} = \underline{\underline{0}}$.

$g(-3)$ esiste ed è $\neq 0$

iv) $\mathbb{R} \ni x \longmapsto -f(x)+1$



$\mathbb{R} \ni x \longmapsto 2g(x+1)$



v) $\displaystyle\int_{-1}^{3} (f(x)-k)\,dx = \int_{-1}^{3} \left(2\sqrt{x+1} - k\right) dx = \int_{-1}^{3} \left(2(x+1)^{1/2} - k\right) dx =$

$\displaystyle = \left[2\,\frac{(x+1)^{3/2}}{3/2} - kx\right]_{-1}^{3} = \left[\frac{4}{3}\left(4^{3/2}\right) - k(3+1)\right] = \frac{4}{3}\cdot 8 - 4k$

$= \dfrac{32}{3} - 4k$;

Ora $\dfrac{32}{3} - 4k = -\dfrac{4}{3} \iff -4k = -\dfrac{36}{3} \iff \underline{k = 3}$

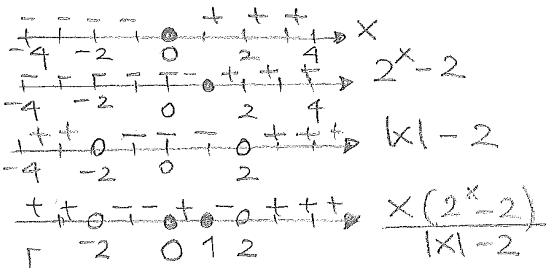
3) • $\log_{1/3} |x^2-2x| \ge -1 \iff \log_{1/3}|x^2-2x| \ge \log_{1/3} 3$

$\iff \begin{cases} |x^2-2x| > 0 \\ |x^2-2x| \le 3 \end{cases}$ (altrimenti non è definito il 1° membro) ($\log_{1/3} x \searrow$)

$$\Leftrightarrow \begin{cases} x^2-2x \neq 0 \\ -3 \leq x^2-2x \leq 3 \end{cases} \Rightarrow \begin{cases} x \neq 0,\ x \neq 2 \\ x^2-2x+3 \geq 0 \\ x^2-2x-3 \leq 0 \end{cases}$$

$$\begin{cases} x \neq 0,\ x \neq 2 \\ \mathbb{R} \\ (x-3)(x+1) \leq 0 \end{cases} \Rightarrow \underline{\underline{S = [-1,3] \setminus \{0,2\}.}} \quad \square$$

- $\dfrac{x(2^x-2)}{|x|-2} \geq 0$

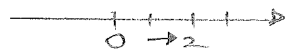


$$\underline{\underline{S =]-\infty,-2[\ \cup\ [0,1]\ \cup\ ]2,+\infty[.}}$$

$\square$

- $\sqrt{x^2-1} \leq 0 \quad \Leftrightarrow \quad x^2-1=0 \qquad \underline{\underline{S=\{-1,1\}.}}$ $\blacksquare$

4) i) • $\displaystyle\lim_{x\to 2^-} \left(\frac{1}{4-x^2} - \frac{1}{x-2}\right) = +\infty - (-\infty)$



$= \underline{\underline{+\infty}}.$

- $\displaystyle\lim_{x\to-\infty} \frac{2^x - x^2}{4x^2+\log|x|} = \underline{\underline{-\frac{1}{4}}},$ poiché $\dfrac{\log|x|}{x^2} \xrightarrow[x\to-\infty]{} 0$

$2^x \xrightarrow[x\to-\infty]{} 0$. $\square$

ii) • $\displaystyle\int_0^1 \frac{3e^x}{e^x+2}\,dx = 3\int_0^1 \frac{e^x}{e^x+2}\,dx = 3\left[\log(e^x+2)\right]_0^1 =$

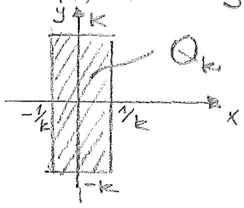
$$= 3\left[\log(e+2) - \log 3\right] = \underline{\underline{3\log\left(\frac{e+2}{3}\right)}}.$$

- $\displaystyle\int_{-2}^{1} \big||x^2+1|-2\big|\,dx = \int_{-2}^{1} |x^2+1-2|\,dx = \int_{-2}^{1} |x^2-1|\,dx =$

$$= \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^{1} (-x^2+1)\,dx = \left[\frac{x^3}{3}-x\right]_{-2}^{-1} + \left[-\frac{x^3}{3}+x\right]_{-1}^{1} =$$

$$= \left(-\frac{1}{3}+1\right) - \left(-\frac{8}{3}+2\right) + \left(-\frac{1}{3}+1\right) - \left(\frac{1}{3}-1\right) = \frac{5}{3}+1 = \underline{\underline{\frac{8}{3}}} \quad \square$$

iii) $\sum_{k=1}^{4}$ perimetro(Q_k) $\qquad Q_k=\{(x,y)\in\mathbb{R}^2: |x|\le \frac{1}{k},\ |y|\le k\}$



$$=\sum_{k=1}^{4}\left(\frac{4}{k}+4k\right)=$$

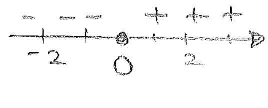
$$=4\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+4(1+2+3+4)=$$

$$=47+\frac{4}{3}=\underline{\frac{145}{3}}$$

5) i) $f(x)=\sqrt{2}\,x\,e^{-x^2+\frac{1}{2}}$

• dom $f=\mathbb{R}=\,]-\infty,+\infty[$ $\qquad f$ è dispari!

• segno f



• $\lim_{x\to-\infty} f(x)=\lim_{x\to+\infty} f(x)=0$ $\Rightarrow$ $y=0$ è asintoto orizzontale per f, per $x\to-\infty$ (per $x\to+\infty$)

• f è continua su tutto $\mathbb{R}$;

• dom $f'=\mathbb{R}$, $\quad f'(x)=\sqrt{2}\,e^{-x^2+\frac{1}{2}}+\sqrt{2}\,x\,e^{-x^2+\frac{1}{2}}(-2x)$

$$=\sqrt{2}\,e^{-x^2+\frac{1}{2}}\left[1-2x^2\right]$$

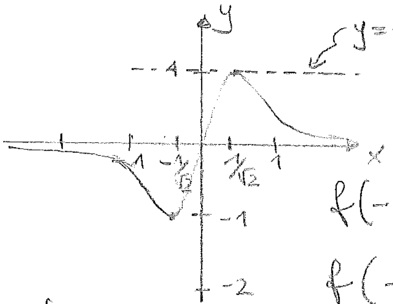
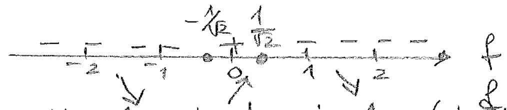


grafico qualitativo di f

$f'(x)=0 \iff x=\pm\frac{1}{\sqrt{2}}$ pt. critici per f



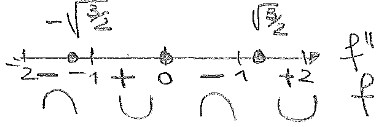
$f\left(-\frac{1}{\sqrt{2}}\right)=-1$

$f\left(\frac{1}{\sqrt{2}}\right)=1$

$x=-1/\sqrt{2}$ pt. di min. loc. (stretto) per f

$x=1/\sqrt{2}$ pt. di max. loc. (stretto) per f

• dom $f''=\mathbb{R}$ $\quad f''(x)=\sqrt{2}\,e^{-x^2+\frac{1}{2}}(2x)(2x^2-3)$

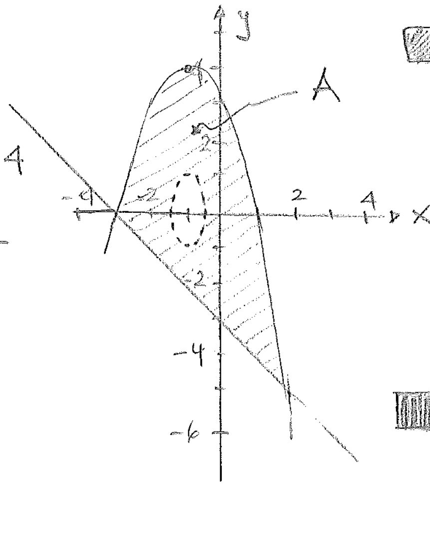


ii) $\underline{y=1}$ eq. della retta tg. al grafico di f nel pt. di coordinata $x=\frac{1}{\sqrt{2}}$ □

iii) $\int_0^{1/\sqrt{2}} f(x)\,dx=\left[-\frac{1}{\sqrt{2}}\,e^{-x^2+\frac{1}{2}}\right]_0^{1/\sqrt{2}}=\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,e^{\frac{1}{2}}\right)$

$$=\underline{\frac{1}{\sqrt{2}}\left(\sqrt{e}-1\right)}.$$

6) $\begin{cases} y\le -x^2-2x+3\\ 4(x^2+2x)+y^2+3>0\\ y\ge -x-3\end{cases} \iff \begin{cases} y\le -(x+1)^2+4\\ \frac{(x+1)^2}{1/4}+y^2>1\\ y\ge -x-3\end{cases}$



FILA (B)

1) i) $\mathcal{P}$ = "$\exists x_1, x_2 \in \mathbb{R}$: $[x_1 \neq x_2$ e $|x_1-2| = |x_2+2|]$"

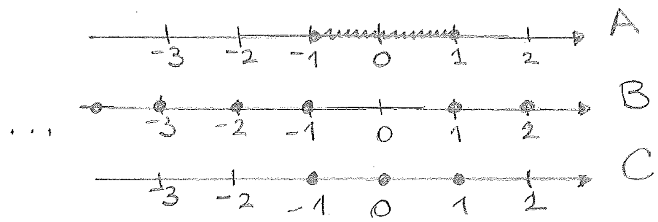
non $\mathcal{P}$ = "$\forall x_1, x_2 \in \mathbb{R}$, $x_1 = x_2$ o $|x_1-2| \neq |x_2+2|$".

Osserviamo che $\mathcal{P}$ è la proposizione vera: infatti, basta prendere $x_1 = 2$ e $x_2 = -2$ (ma qualsiasi altra coppia $x_1 = x$, $x_2 = -x$ con $x \in \mathbb{R}$ va bene). $\square$

ii) $A = [-1, 1]$,

$B = \{x \in \mathbb{Z} : x^2 \geq 1\} = \{\ldots, -4, -3, -2, -1, 1, 2, 3, \ldots\}$

$C = \{-1, 0, 1\}$



$[0,1] \subset A$ (V) poiché $\forall x \in [0,1]$ si ha $x \in A$.

$(-1,-1) \in B \times A$ (V) poiché $-1 \in B$, $-1 \in A$.

$\{0,1\} \in \mathcal{P}(C)$ (V) poiché $\{0,1\} \subset C$.

$]0,1[\in \mathcal{P}(B)$ (F) poiché $]0,1[\subset \mathbb{R} \setminus B$.

$A \times \{1\} \subseteq A \times B$ (V) poiché $\{1\} \subset B$.

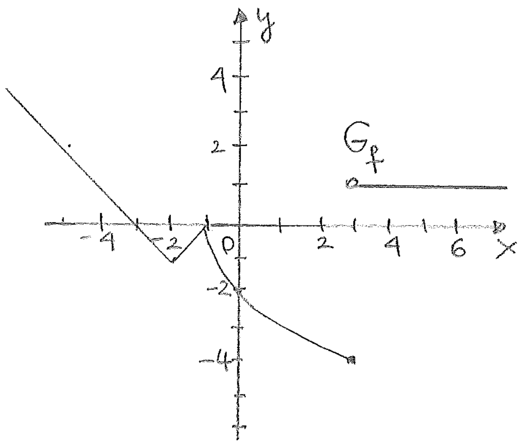
$\mathbb{R} \setminus A = B$ (F) poiché $1 \in B$ ma $1 \notin \mathbb{R} \setminus A$.

$C \subset A$ (V) poiché ogni $x \in C$ appartiene anche ad A.

$-2 \in B$ (V) $\blacksquare$

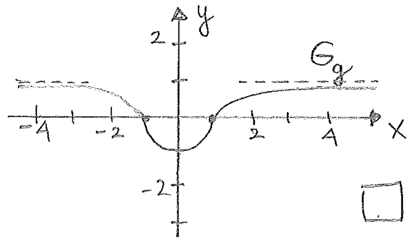
2) $f : \mathbb{R} \to \mathbb{R}$

i) $$f(x) = \begin{cases} |x+2| - 1 & \text{se } x < -1 \\ -2\sqrt{x+1} & \text{se } -1 \leq x \leq 3 \\ 1 & \text{se } x > 3 \end{cases}$$



$g: \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} x^2 - 1 & \text{se } -1 \le x \le 1 \\ -\frac{1}{x^2} + 1 & \text{se } x < -1 \text{ o } x > 1 \end{cases}$$



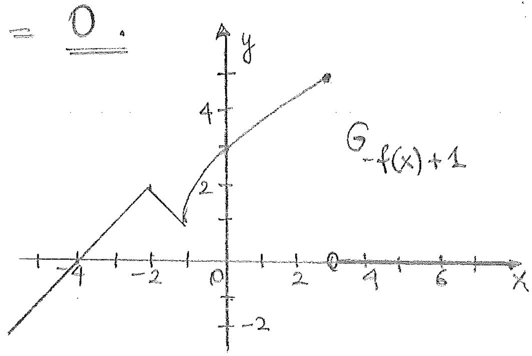
ii) $\underline{f(\mathbb{R}) = [-4, +\infty[}$. f è limitata inferiormente, ma non è limitata superiormente, quindi $\underline{f \text{ non è limitata}}$.

iii) $(g \circ f)(-2) \underset{\operatorname{dom} g = \mathbb{R} \supset f(\mathbb{R})}{=} g(f(-2)) = g(-1) = \underline{\underline{0}}$;

$(f - g)(0) = f(0) - g(0) = -2 - (-1) = \underline{\underline{-1}}$;

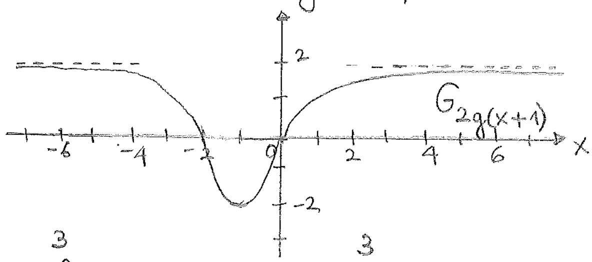
$\left(\frac{f}{g}\right)(-3) = \frac{f(-3)}{g(-3)} = \frac{0}{g(-3)} = \underline{\underline{0}}$.

$g(-3)$ ∃ ed è $\neq 0$



iv) $\mathbb{R} \ni x \mapsto -f(x) + 1$

$\mathbb{R} \ni x \mapsto 2g(x+1)$



v) $$\int_{-1}^{3} (f(x) + k)\,dx = \int_{-1}^{3} \left(-2\sqrt{x+1} + k\right) dx = \int_{-1}^{3} \left(-2(x+1)^{1/2} + k\right) dx$$

$$= \left[-2\,\frac{(x+1)^{3/2}}{3/2} + kx\right]_{-1}^{3} = \left[-\frac{4}{3}\left(4^{3/2}\right) + k(3+1)\right] = -\frac{4 \cdot 8}{3} + 4k$$

$$= -\frac{32}{3} + 4k$$

Ora $\int_{-1}^{3} (f(x) + k)\,dx = \frac{4}{3} \iff \frac{36}{3} = 4k \iff \underline{\underline{k = 3}}$.

3) • $\log_{1/4} |x^2 - 3x| \ge -1 \iff \log_{1/4} |x^2 - 3x| \ge \log_{1/4} 4$

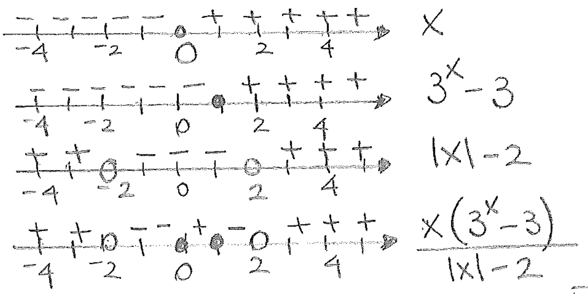
$$\iff \begin{cases} |x^2 - 3x| > 0 & \text{(altrimenti non è definito il logaritmo)} \\ |x^2 - 3x| \le 4 & (\log_{1/4} x \searrow) \end{cases}$$

$$\Leftrightarrow \begin{cases} x^2-3x \neq 0 \\ -4 \leq x^2-3x \leq 4 \end{cases} \Rightarrow \begin{cases} x \neq 0 \quad x \neq 3 \\ x^2-3x+4 \geq 0 \\ x^2-3x-4 \leq 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \neq 0,\ x \neq 3 \\ \mathbb{R} \\ (x-4)(x+1) \leq 0 \end{cases} \Rightarrow \underline{\underline{S = [-1,4] \setminus \{0,3\}}}$$

$\square$

- $$\frac{x(3^x-3)}{|x|-2} \leq 0$$



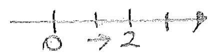
$$\underline{\underline{S =]-2,0] \cup [1,2[}}.$$

$\square$

- $\sqrt{x^2-4} \leq 0 \quad \Leftrightarrow \quad x^2-4=0 \qquad \underline{\underline{S=\{-2,2\}}}.$

$\blacksquare$

4) i) • $$\lim_{x\to 2^-}\left(\frac{1}{x^2-4}+\frac{1}{x-2}\right) = -\infty-\infty = \underline{\underline{-\infty}}$$



- $$\lim_{x\to -\infty} \frac{x^2+3^x}{-2x^2+\log|x|} = \underline{\underline{-\frac{1}{2}}},$$ poiché $\frac{\log|x|}{x^2} \xrightarrow[x\to-\infty]{} 0$.

$3^x \xrightarrow[x\to-\infty]{} 0$ (con $3^x \to 0$)

$\square$

ii) • $$\int_0^1 \frac{2e^x}{e^x+3}\,dx = 2\int_0^1 \frac{e^x}{e^x+3}\,dx = 2\Big[\log(e^x+3)\Big]_0^1 =$$

$$= 2\Big[\log(e+3)-\log 4\Big] = \underline{\underline{2\log\left(\frac{e+3}{4}\right)}}.$$

- $$\int_{-4}^{2} \big||x^2+2|-6\big|\,dx = \int_{-4}^{2}|x^2+2-6|\,dx = \int_{-4}^{2}|x^2-4|\,dx =$$

$$= \int_{-4}^{-2}(x^2-4)\,dx + \int_{-2}^{2}(-x^2+4)\,dx = \left[\frac{x^3}{3}-4x\right]_{-4}^{-2} + \left[-\frac{x^3}{3}+4x\right]_{-2}^{2}$$

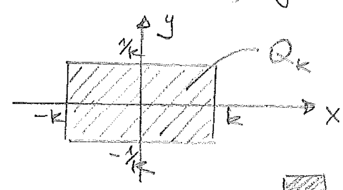
$$= \left(-\frac{8}{3}+\cancel{8}\right)-\left(-\frac{64}{3}+\cancel{16}\right)+\left(-\frac{8}{3}+\cancel{8}\right)-\left(+\frac{8}{3}-8\right) =$$

$$= \frac{64}{3}-\cancel{\frac{24}{3}}+\cancel{\frac{24}{3}} = \underline{\underline{\frac{64}{3}}}.$$

$\square$

iii) $\sum_{k=1}^{4}$ perimetro (Q_k) $\qquad Q_k=\left\{(x,y)\in\mathbb{R}^2 : |x|\le k,\ |y|\le \frac{1}{k}\right\}$

$$= \sum_{k=1}^{4}\left(\frac{4}{k}+4k\right)=\frac{145}{3}.$$



5) i) La funzione $f(x)=-\sqrt{2}\,x\,e^{-x^2+1/2}$ è esattamente la funzione studiata in FILA (A), Es. 5 i) cambiata di segno.

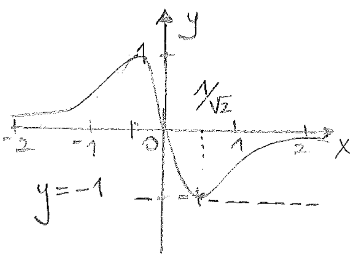


grafico qualitativo di f.

ii) $\underline{y=-1}$ eq. della retta tg. al grafico di f nel pt. di coordinata $x=\frac{1}{\sqrt{2}}$.

iii) $$\int_{-1/\sqrt{2}}^{0} f(x)\,dx=\int_{-1/\sqrt{2}}^{0} -\sqrt{2}\,x\,e^{-x^2+1/2}\,dx=$$

$$=\left[\frac{1}{\sqrt{2}}\,e^{-x^2+1/2}\right]_{-1/\sqrt{2}}^{0}=\frac{1}{\sqrt{2}}\,e^{1/2}-\frac{1}{\sqrt{2}}=\underline{\frac{1}{\sqrt{2}}\left(\sqrt{e}-1\right)}.$$

6) $$\begin{cases} y\ge x^2+2x-3\\ 4(x^2+2x)+y^2+3>0\\ y\le x+3\end{cases}\iff\begin{cases} y\ge (x+1)^2-4\\ \frac{(x+1)^2}{1/4}+y^2>1\\ y\le x+3\end{cases}$$