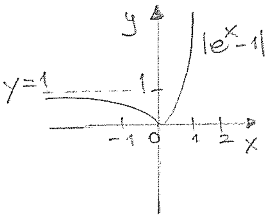
Università di Trento – Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
SECONDA PROVA INTERMEDIA DI ANALISI MATEMATICA
a.a. 2013–2014 – Rovereto, 23 gennaio 2014

FILA (A)

1) i) • $\displaystyle\int_2^3 \frac{3x^2+x^{-1}+x}{\sqrt{x}}\,dx = \int_2^3 \left(3x^{3/2}+x^{-3/2}+x^{1/2}\right)dx = \left[3\frac{x^{5/2}}{5/2}+\frac{x^{-1/2}}{-1/2}+\frac{x^{3/2}}{3/2}\right]_2^3$

$$= \left[\frac{6}{5}x^{5/2}-\frac{2}{\sqrt{x}}+\frac{2}{3}x^{3/2}\right]_2^3 = \left(\frac{6}{5}3^{5/2}-\frac{2}{\sqrt{3}}+\frac{2}{3}3^{3/2}\right)-\left(\frac{6}{5}2^{5/2}-\frac{2}{\sqrt{2}}+\frac{2}{3}2^{3/2}\right)$$

$$= \left(\frac{54}{5}\sqrt{3}-\frac{2\sqrt{3}}{3}+2\sqrt{3}\right)-\left(\frac{24}{5}\sqrt{2}-\sqrt{2}+\frac{4}{3}\sqrt{2}\right) = \underline{\frac{152}{15}\sqrt{3}-\frac{77}{15}\sqrt{2}}$$

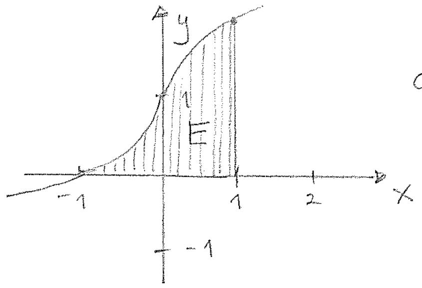


• $\displaystyle\int_{-1}^2 |e^x-1|\,dx = \int_{-1}^0(-e^x+1)\,dx+\int_0^2(e^x-1)\,dx = \left[-e^x+x\right]_{-1}^0+\left[e^x-x\right]_0^2 =$

$$= \left[(-1)-(-e^{-1}-1)\right]+\left[(e^2-2)-(1)\right] = \underline{\frac{1}{e}+e^2-3}\,.$$

• $\displaystyle\lim_{x\to-1^-}\Bigg(\underbrace{\frac{1}{x+1}}_{\to-\infty}-\underbrace{\frac{1}{x^2-1}}_{\to+\infty}\Bigg) = \underline{\underline{-\infty}}\,.$

ii)



$$\text{area}(E) = \int_{-1}^1\left(\sqrt[3]{x}+1\right)dx = \int_{-1}^1\left(x^{1/3}+1\right)dx =$$

$$= \left[\frac{3}{4}x^{4/3}+x\right]_{-1}^1 = \left[\left(\frac{3}{4}(-1)^{4/3}+1\right)-\left(\frac{3}{4}(-1)^{4/3}-1\right)\right]$$

$$= \underline{\underline{2}}\ ^{(*)}$$

iii) $\displaystyle\sum_{n=1}^5 f\left((-1)^n\frac{1}{n}\right)$, dove $f(x)=\begin{cases}|x-1| & \text{se } x\le 0\\ \log x & \text{se } x>0\end{cases}$.

$$\sum_{n=1}^5 f\left((-1)^n\frac{1}{n}\right) = f(-1)+f\left(\frac{1}{2}\right)+f\left(-\frac{1}{3}\right)+f\left(\frac{1}{4}\right)+f\left(-\frac{1}{5}\right) =$$

$$= |-2|+\log\frac{1}{2}+\left|-\frac{1}{3}-1\right|+\log\frac{1}{4}+\left|-\frac{1}{5}-1\right| =$$

$$= 2+\frac{4}{3}+\frac{6}{5}-\log 2-\log 4 = \frac{68}{15}-\log 8\,.$$

iv) $\displaystyle\frac{1}{2}-3+\frac{1}{4}-6+\ldots+\frac{1}{20}-30 = \sum_{n=1}^{10}\left(\frac{1}{2n}-3n\right)$.

(*) A questo risultato si raggiunge immediatamente calcolando l'area del triangolo T di vertici $(-1,0)$, $(1,0)$ e $(1,2)$.

2) i) $f : [-1,1] \to \mathbb{R}$

$$f(x) = \begin{cases} -x^2 - bx + c & \text{se } -1 \le x < 0 \\ d & \text{se } x = 0 \\ 2e^x & \text{se } 0 < x \le x \end{cases}$$

è continua in $x=0$ $\iff$

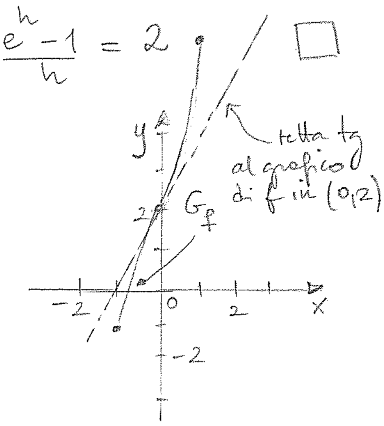
$$\begin{cases} \lim\limits_{x\to 0^-} (-x^2 - bx + c) = \lim\limits_{x \to 0^+} 2e^x = \\ = f(0) = d \end{cases}$$

$\iff$ $\underline{c = 2 = d}$.

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h^2 - bh + \cancel{2} - \cancel{2}}{h} = -b$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad \underline{b = -2}$

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{2e^h - 2}{h} = 2 \lim_{h\to 0} \frac{e^h - 1}{h} = 2$$



ii)
$$f(x) = \begin{cases} -x^2 + 2x + 2 & \text{se } -1 \le x \le 0 \\ 2e^x & \text{se } 0 < x \le 1 \end{cases}$$

$$-x^2 + 2x + 2 = -(x^2 - 2x) + 2$$
$$= -(x-1)^2 + 3$$

$$-x^2 + 2x + 2 = 0 \iff x_{1/2} = \frac{-2 \pm \sqrt{4+8}}{-2} = \frac{-2 \pm 2\sqrt{3}}{-2}$$
$$= 1 \pm \sqrt{3}$$

iii) $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = 2$ (vedi pt. i)) ; quindi

$y = 2 + 2(x-0)$ è l'eq. della retta tg. al grafico di f nel pt. di coordinate $x=0$, ossia $\underline{y = 2x+2}$.

iv) $\min\limits_{[-1,1]} f = \underline{\underline{-1}}$ $\qquad$ $\underline{\underline{x = -1}}$ pt. di minimo di f su $[-1,1]$.

$\max\limits_{[-1,1]} f = \underline{\underline{2e}}$ $\qquad$ $\underline{\underline{x = 1}}$ pt. di massimo di f su $[-1,1]$.

v)
$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} (-x^2 + 2x + 2)\,dx + \int_{0}^{1} 2e^x\,dx = \left[-\frac{x^3}{3} + x^2 + 2x\right]_{-1}^{0} + \left[2e^x\right]_0^1$$

$$= -\left(\frac{1}{3} + 1 - 2\right) + (2e - 2) = \underline{\underline{2e - \frac{4}{3}}}.$$

3) • $\log_2 (x^2 - 4) + \log_{1/2} (x+2) \le 3 \iff$

$\log_2 (x^2-4) - \log_2 (x+2) \le \log_2 8$

$$\iff \begin{cases} x^2-4>0 \\ x+2>0 \\ \dfrac{x^2-4}{x+2} \le 8 \end{cases} \iff \begin{cases} x<-2 \text{ o } x>2 \\ x>-2 \\ \dfrac{(x-2)\cancel{(x+2)}}{\cancel{x+2}} \le 8 \end{cases} \qquad \underline{\underline{S =]2,10]}}.$$

□

• $$9^{x^2} \cdot 3^{|x^2-1|} < 3^{\log_3 9} \iff 3^{2x^2+|x^2-1|} < 3^2$$

$$\iff 2x^2+|x^2-1|<2 \iff \begin{cases} x^2-1 \ge 0 \\ 3x^2<3 \end{cases} \text{ o } \begin{cases} x^2-1<0 \\ x^2<1 \end{cases}$$

$$\iff \begin{cases} \emptyset \quad \text{o} \quad x^2<1 \end{cases} \Rightarrow \underline{\underline{S=]-1,1[}}.$$

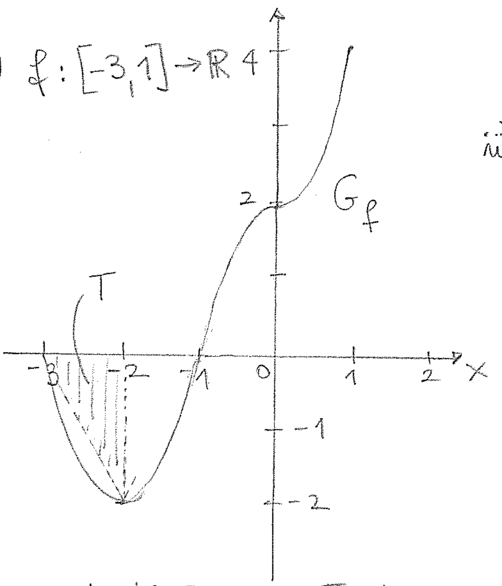
□

• $$\left(\int_{-1}^{x} e^{-t^2}\,dt\right)' = \frac{1}{e^4} \underset{TFC}{\iff} e^{-x^2} = e^{-4} \iff -x^2=-4$$

$$\Rightarrow \underline{\underline{S=\{-2,2\}}}$$

■

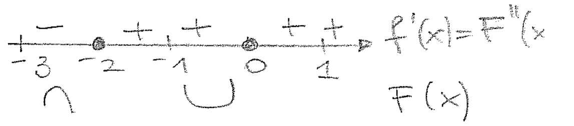
4) $f:[-3,1]\to\mathbb{R}$

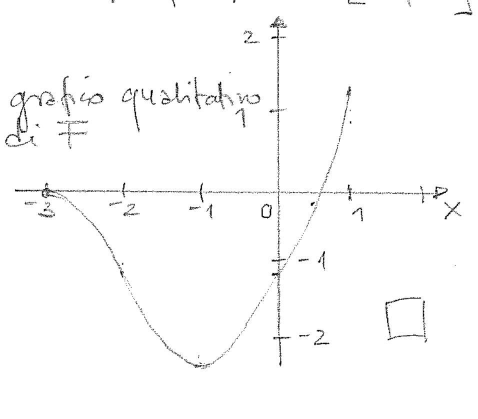


ii) Dalla definizione di $F(x)=\int_{-3}^{x} f(t)\,dt$ e dalla interpretazione geom. dell'integrale si ottiene immediatamente che F ristretta a $[-3,-1]$ è decrescente, mentre è crescente su $[-1,1]$.

Tale risultato si può anche ottenere usando il Teorema Fondamentale del Calcolo integrale che asserisce che $F'(x)=f(x)$ $\forall x\in[-3,1]$. Dal segno di f si ottiene quindi il segno di $F'(x)$ e quindi si determina la monotonia di $F(x)$ su $[-3,1]$. □

ii) Essendo f derivabile su $[-3,1]$, da $F'(x)=f(x)$ su $[-3,1]$ si ottiene $F''(x)=f'(x)$ su $[-3,1]$ e quindi





Sia $E_1=\{(x,y)\in\mathbb{R}^2: -3\le x\le 2,\ f(x)\le y\le 0\}$

$$F(0) = -2\,\text{area}(E_1) + \text{area}(E_1) = -\text{area}(E_1)$$

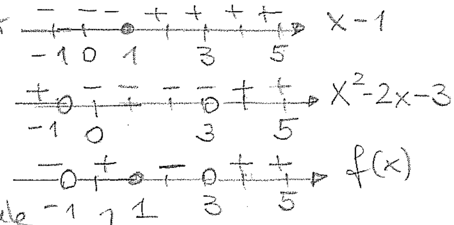
$$< -\text{area}(T) = -1$$

□

■

5) i) $f(x)=\dfrac{x-1}{x^2-2x-3}=\dfrac{x-1}{(x-3)(x+1)}$ • $\operatorname{dom} f=\underline{\mathbb{R}\setminus\{-1,3\}}$

• segno di f



• $\lim\limits_{x\to-\infty\,(+\infty)} f(x)=0$ $y=0$ asintoto orizz. per f, per $x\to-\infty\ (+\infty)$;

$\lim\limits_{x\to-1^-} f(x)=-\infty$ $x=-1$ asintoto verticale per f;

$\lim\limits_{x\to-1^+} f(x)=+\infty$ $x=-1$ asintoto verticale per f;

$\lim\limits_{x\to3^-} f(x)=-\infty$ $x=3$ asintoto verticale per f;

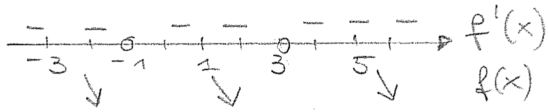
$\lim\limits_{x\to3^+} f(x)=+\infty$ $x=3$ asintoto verticale per f.

per calcolare questi limiti si deve usare il fatto che $f(x)=\dfrac{x-1}{(x-3)(x+1)}$

• f è continua su $\mathbb{R}\setminus\{-1,3\}$.

• f è derivabile su $\mathbb{R}\setminus\{-1,3\}$ e $f'(x)=\dfrac{x^2-2x-3-(x-1)(2x-2)}{(x^2-2x-3)^2}=$

$$=\frac{x^2-2x-3-2x^2+4x-2}{(x^2-2x-3)^2}=\frac{-x^2+2x-5}{(x^2-2x-3)^2}.$$



ii) $\operatorname{dom} f''=\mathbb{R}\setminus\{-1,3\}$

$$f''(x)=\frac{(-2x+2)(x^2-2x-3)^{\cancel{2}}-(-x^2+2x-5)\,2\,\cancel{(x^2-2x-3)}(2x-2)}{(x^2-2x-3)^{\cancel{4}\,3}}$$

$$=\frac{-2x^3+\cancel{4x^2}+6x+2x^2-4x-6+4x^3-\cancel{4x^2}-8x^2+8x+20x-\ldots}{(x^2-2x-3)^3}$$

$$=\frac{2x^3-6x^2+30x-26}{(x^2-2x-3)^3}=\frac{(x-1)(\overbrace{2x^2-4x}^{10}+26)}{(x^2-2x-3)^3}$$

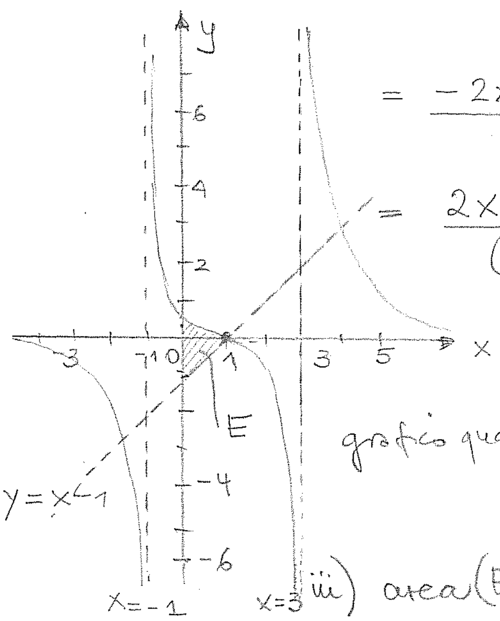
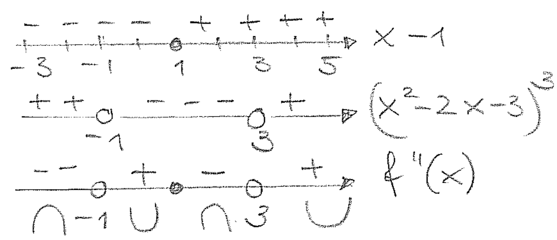


grafico qualitativo di f



iii) $\operatorname{area}(E)=\displaystyle\int_0^1 (f(x)-x+1)\,dx=\left[\frac{1}{2}\log|x^2-2x-3|\right]_0^1+\left[-\frac{x^2}{2}+x\right]_0^1=$

$$=\frac{1}{2}\left(\log|1-5|-\log|-3|\right)+\left[\left(-\frac{1}{2}+1\right)\right]=\underline{\underline{\frac{1}{2}\left(\log\frac{4}{3}+1\right)}}.$$

■

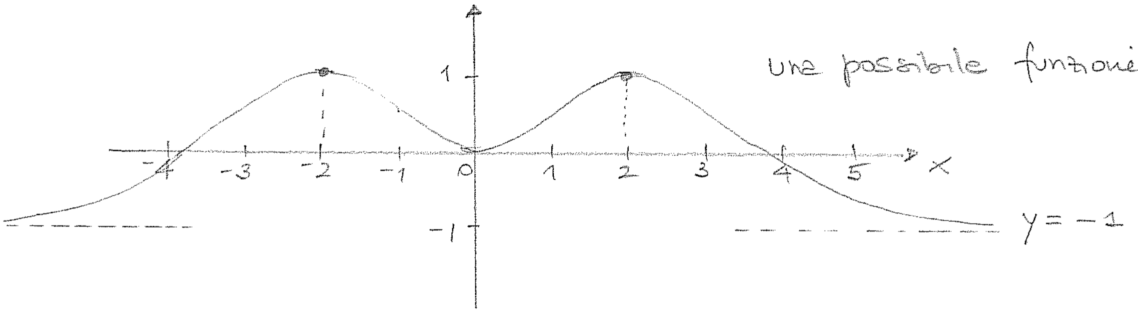
6) $f: \mathbb{R} \to \mathbb{R}$ continua e derivabile:

f è pari, ossia $f(x) = f(-x)\ \forall x \in \mathbb{R}$

$f(2) = 1$ (e quindi $f(-2) = 1$)

$f'(2) = 0$ (" $f'(-2) = 0$)

$\lim_{x \to +\infty} f(x) = -1$ $\left(\lim_{x \to -\infty} f(x) = -1 \right)$



una possibile funzione f