Università di Trento – Dip. di Psicologia e Scienze Cognitive
CdL in SCIENZE E TECNICHE DI PSICOLOGIA COGNITIVA
ESAME SCRITTO DI ANALISI MATEMATICA
a.a. 2013 – 2014 — Rovereto, 13 febbraio 2014

<u>FILA (A)</u>

1) i) $\mathcal{A} =$ " $\forall x_1, x_2 \in [0,1]$, $[(x_1 < x_2) \Rightarrow (-3x_1 < 2x_2)]$ ".

non $\mathcal{A} =$ " $\exists x_1, x_2 \in [0,1] : [(x_1 < x_2)$ e $(-3x_1 \geq 2x_2)]$ ".

La proposizione $\mathcal{A}$ è vera, poiché $\forall x_1, x_2 \in [0,1]$, $x_1 < x_2$ si ha $-3x_1 \leq 0$, mentre $2x_1 > 0$, quindi $-3x_1 < 2x_2$. $\square$

ii) $$\lim_{x \to -1^+} \frac{1}{\underbrace{|x^3| - 1}_{\text{infinitesimo negativo}}} = \underline{\underline{-\infty}}.$$

$$\int_0^1 \frac{1 - e^{3x}}{e^x}\, dx = \int_0^1 \left[e^{-x} - e^{2x}\right] dx = -\int_0^1 \left[-e^{-x}\right] dx - \frac{1}{2}\int_0^1 2e^{2x} =$$

$$-\left[e^{-x}\right]_0^1 - \frac{1}{2}\left[e^{2x}\right]_0^1 = -\left(e^{-1} - 1\right) - \frac{1}{2}\left(e^2 - 1\right) = \underline{\underline{-\frac{1}{2}e^2 - \frac{1}{e} + \frac{3}{2}}}.$$

$$\sum_{k=1}^{20} \left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right) = \left(1 - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}}\right) + \dots + \left(\cancel{\frac{1}{19^2}} - \cancel{\frac{1}{20^2}}\right) + \left(\cancel{\frac{1}{20^2}} - \frac{1}{21^2}\right)$$

$$= \underline{\underline{1 - \frac{1}{21^2}}}.$$ ■

2) $\mathcal{E}$: $x^2 - 4x + 4y^2 - 8y + 4 = 0$

$(x-2)^2 - 4 + 4(y-1)^2 - \cancel{4} + \cancel{4} = 0$

$(x-2)^2 + 4(y-1)^2 = 4$

$\dfrac{(x-2)^2}{4} + (y-1)^2 = 1$ $\qquad C = (2,1)$ $\quad a = 2$, $b = 1$.

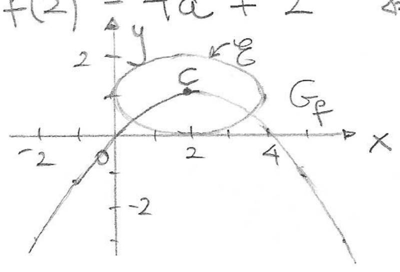
i) $f(x) = ax^2 + bx + c$ $\qquad f'(x) = 2ax + b$; allora

a) $f'(0) = 1 \iff b = 1$;

b) $(0,0) \in G_f \iff 0 = f(0) = c$;

$(2,1) \in G_f \iff 1 = f(2) = 4a + 2 \iff a = -\frac{1}{4}$. $\square$

ii) $\underline{\underline{f(x) = -\frac{1}{4}x^2 + x}}$ $\square$



iii) $A = [2, +\infty[\ \subset [0, +\infty[$ e $f|_A$ è iniettiva. $\square$

iv) $f(x) = -\frac{1}{4}x^2 + x \qquad y = -x$. Determiniamo i pt. di intersezione della retta di eq. $y=-x$ con il grafico di f:

$$\begin{cases} y = -x \\ -\frac{1}{4}x^2 + x = -x \end{cases} \Longleftrightarrow \begin{cases} y = -x \\ \frac{1}{4}x^2 - 2x = 0 \end{cases} \Longleftrightarrow \begin{cases} y = -x \\ x(x-8) = 0 \end{cases} \quad \begin{matrix} S_1 = (0,0) \\ S_2 = (8,-8) \end{matrix}$$

Allora l'area della regione piana delimitata dal grafico di f e dalla retta di eq. $y=-x$ è data da

$$\int_0^8 \left(-\frac{1}{4}x^2 + x + x\right) dx = \int_0^8 \left(-\frac{1}{4}x^2 + 2x\right) dx = \left[-\frac{1}{4}\frac{x^3}{3} + x^2\right]_0^8 =$$

$$= -\frac{1}{\cancel{4}}\frac{\cancel{8}^{\,2}\cdot 64}{3} + 64 = \underline{\underline{\frac{64}{3}}}.$$ $\blacksquare$

3) i) $A = \{x \in \mathbb{R} : \log_{\frac{1}{2}}(1+x) + \log_{\frac{1}{2}}(2x) \geq -2\}$

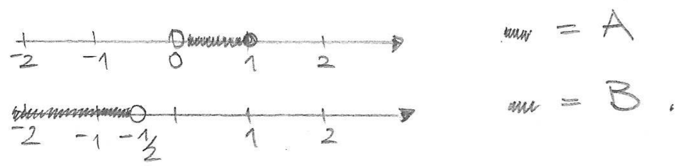
$$\begin{cases} 1+x>0 \\ x>0 \\ (1+x)\cdot 2x \leq 4 \end{cases} \Longleftrightarrow \begin{cases} x>-1 \\ x>0 \\ 2x^2+2x-4 \leq 0 \end{cases} \Longleftrightarrow \begin{cases} x>-1 \\ x>0 \\ (x+2)(x-1) \leq 0 \end{cases}$$

$$\begin{cases} x>-1 \\ x>0 \\ -2 \leq x \leq 1 \end{cases} \Rightarrow \underline{\underline{A =]0,1]}}.$$

$B = \left\{x \in \mathbb{R} : \frac{x+|x+1|}{2^x} < 0\right\}$. Poiché $2^x > 0 \ \forall x \in \mathbb{R}$, basta determinare gli $x \in \mathbb{R}$ t.c. $x + |x+1| < 0$. Si ha

$$\begin{cases} x+1 \geq 0 \\ 2x+1 < 0 \end{cases} \text{ o } \begin{cases} x+1<0 \\ -1<0 \end{cases} \Longleftrightarrow$$

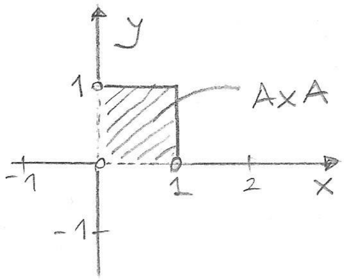
$$\begin{cases} x \geq -1 \\ x < -\frac{1}{2} \end{cases} \text{ o } \begin{cases} x<-1 \\ \mathbb{R} \end{cases} \Rightarrow \underline{\underline{B =]-\infty, -\tfrac{1}{2}[}}.$$ $\square$



$\max A = 1 \quad \nexists \min A \ ; \quad \nexists$ sono $\max B$, $\min B$. $\square$

ii) $A \cup B = \underline{]-\infty, -\frac{1}{2}[\cup]0,1]}$; non è un intervallo.

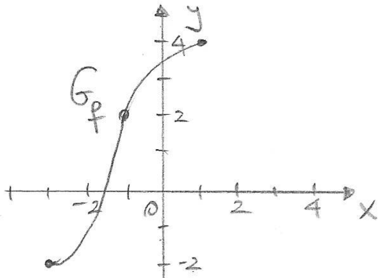
iii) $A \times A$

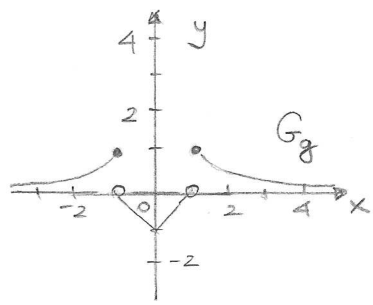


4) $f : [-3,1] \to \mathbb{R}$ $\qquad$ $g : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} (x+3)^2 - 2 & \text{se } -3 \le x < -1 \\ 2\log_2(x+3) & \text{se } -1 \le x \le 1 \end{cases} \qquad g(x) = \begin{cases} |x| - 1 & \text{se } -1 < x < 1 \\ \frac{1}{x^2} & \text{se } x \in \mathbb{R} \setminus]-1,1[. \end{cases}$$

i)





ii) $f([-3,1]) = \underline{[-2,4]}$.

$f^{-1}(-1) = x \iff f(x) = -1$; risulta $\underline{x = -2}$

$f^{-1}(4) = x \iff f(x) = 4$; risulta $\underline{x = 1}$.

iii) $(f \circ g)(x)$ per $x \in]-1,1[$: osserviamo che

$g(]-1,1[) = [-1,0[$; ora $f(x) = 2\log_2(x+3)$ per $x \in [-1,1]$, quindi

$$(f \circ g)(x) \underset{x \in]-1,1[}{\doteq} f(g(x)) = 2\log_2(g(x)+3) = 2\log_2(|x| - 1 + 3) =$$
$$= \underline{2\log_2(|x|+2)}.$$

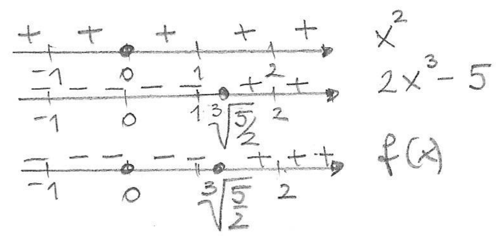
iv) g non soddisfa le ipotesi del teorema di Weierstrass su $[-2,2]$, non essendo continua in $x = -1$, $x = 1$

(per es. $\lim_{x \to -1^-} \frac{1}{x^2} = 1 \neq \lim_{x \to -1^+} (|x| - 1) = 0$).

v) $\nexists \min_{\mathbb{R}} |g(x)|$ $\qquad$ $\max_{\mathbb{R}} |g(x)| = 1$ $\qquad$ $x \in \{-1, 0, 1\}$ sono i pt. di massimo

5) $f(x) = 2x^5 - 5x^2$

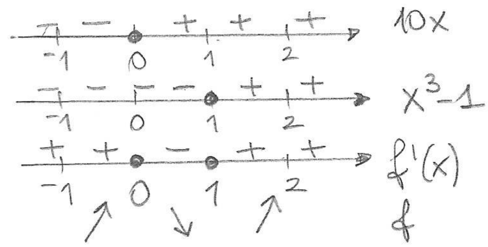
- $\text{dom} f = \mathbb{R} =]-\infty, +\infty[$
- segno di f: $f(x) = x^2(2x^3 - 5)$



- $\lim\limits_{x \to -\infty} f(x) = -\infty$

  $\lim\limits_{x \to +\infty} f(x) = +\infty$

- f è continua su $\mathbb{R}$.
- $\text{dom} f' = \mathbb{R}$ $\quad f'(x) = 10x^4 - 10x = 10x(x^3 - 1)$

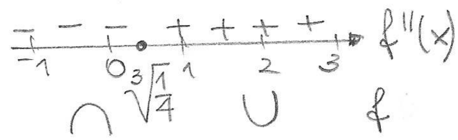


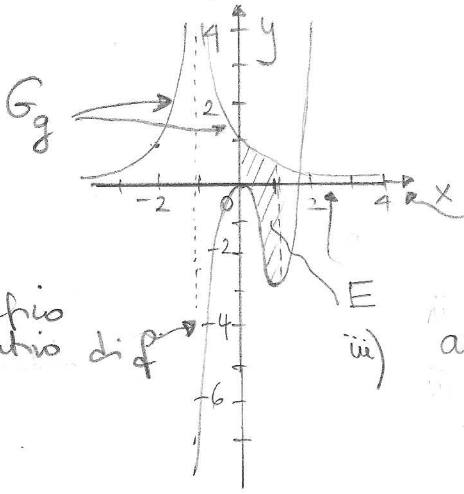
$f'(x) = 0 \iff x = 0,\ x = 1$

$x = 0$ pt di max. loc. stretto per f; e $f(0) = 0$;

$x = 1$ pt. di min. loc. stretto per f; e $f(1) = -3$.

- $\text{dom} f'' = \mathbb{R}$ $\quad f''(x) = 40x^3 - 10 = 10(4x^3 - 1)$





ii) $\underline{\underline{y = 0}}$.

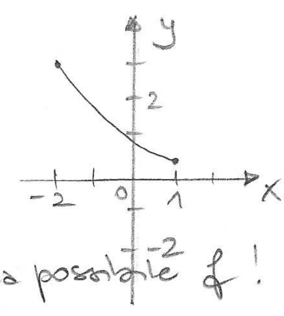
iii) $\text{area}(E) = \int_0^1 \left(\frac{1}{(x+1)^2} - 2x^5 + 5x^2 \right) dx =$

$= \left[-\frac{1}{x+1} - \cancel{2}\frac{x^6}{\cancel{6}_3} + 5\frac{x^3}{3} \right]_0^1 =$

$= \left(-\frac{1}{2} - \frac{1}{3} + \frac{5}{3} \right) - (-1) = \underline{\underline{\frac{11}{6}}}$.

6) $f : [-2, 1] \to \mathbb{R}$

$F(x) = \int_{-2}^{x} f(t)\,dt$; dal teorema fond. del calcolo integrale abbiamo che $f(x) > 0 \Rightarrow F$ crescente, mentre $f'(x) < 0 \Rightarrow F$ concava!



Una possibile f!

FILA (B)

1) i) $\mathcal{A}=$ " $\forall x_1,x_2\in[0,1],\ \ [(x_1<x_2)\Rightarrow(x_1>x_2^2)]$ ".

non $\mathcal{A}=$ " $\exists x_1,x_2\in[0,1] : [(x_1<x_2)$ e $(x_1\le x_2^2)]$ ".

non $\mathcal{A}$ è vera: basta prendere $x_1=\frac{1}{2}$, $x_2=1$. Da questo segue che la proposizione $\mathcal{A}$ è falsa.

ii) $$\lim_{x\to-1^-}\frac{1}{\underbrace{|x^3|-1}_{\text{infinitesimo positivo}}}=\underline{+\infty}.$$

$$\int_0^1\frac{e^{2x}+1}{e^x}\,dx=\int_0^1(e^x+e^{-x})\,dx=\left[e^x\right]_0^1-\left[e^{-x}\right]_0^1=(e-1)-\left(\frac{1}{e}-1\right)$$
$$=\underline{e-\frac{1}{e}}.$$

$$\sum_{k=1}^{21}\left(\frac{1}{(k+1)^2}-\frac{1}{k^2}\right)=\left(\cancel{\frac{1}{4}}-1\right)+\left(\cancel{\frac{1}{9}}-\cancel{\frac{1}{4}}\right)+\left(\frac{1}{16}-\cancel{\frac{1}{9}}\right)+\dots+$$
$$+\left(\cancel{\frac{1}{21^2}}-\frac{1}{20^2}\right)+\left(\frac{1}{22^2}-\cancel{\frac{1}{21^2}}\right)=\underline{-1+\frac{1}{22^2}}.$$

2) $\mathcal{E}:\ x^2+4x+4y^2-8y+4=0$

$$(x+2)^2-4+4(y-1)^2-\cancel{4}+\cancel{4}=0$$
$$\frac{(x+2)^2}{4}+(y-1)^2=1\qquad C=(-2,1)\quad a=2,\ b=1.$$

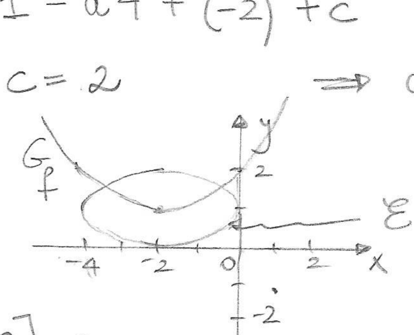
i) $f(x)=ax^2+bx+c\qquad f'(x)=2ax+b$; allora

a) $f'(0)=1\iff b=1$;

b) $(-2,1)\in G_f\iff 1=a4+(-2)+c\Rightarrow 3=4a+c$

$(0,2)\in G_f\iff c=2\Rightarrow a=\frac{1}{4}$.

ii) $\underline{f(x)=\frac{1}{4}x^2+x+2}$



iii) $A=\,]-\infty,-2]\subset\,]-\infty,0]$ e

$f|_A$ è iniettiva.

iv) $f(x)=\frac{1}{4}x^2+x+2\quad y=-x+2$. Determiniamo i pt. di intersezione della retta di eq. $y=-x+2$ con il grafico di f:

$$\begin{cases} y=-x+2 \\ \frac{1}{4}x^2+x+\cancel{2}=-x+\cancel{2} \end{cases} \Leftrightarrow \begin{cases} y=-x+2 \\ x^2+8x=0 \end{cases} \Leftrightarrow \begin{cases} y=-x+2 \\ x(x+8)=0 \end{cases} \qquad \begin{matrix} S_1=(0,2) \\ S_2=(-8,10) \end{matrix}$$

Allora l'area della regione piana delimitata dal grafico di f e dalla retta di eq. $y=-x+2$ è data da

$$\int_{-8}^{0}\left(-x+\cancel{2}-\frac{1}{4}x^2-x-\cancel{2}\right)dx = \int_{-8}^{0}\left(-\frac{1}{4}x^2-2x\right)dx = \left[-\frac{1}{4}\frac{x^3}{3}-x^2\right]_{-8}^{0} =$$

$$= -\left(+\frac{1}{\cancel{4}}\cdot\frac{\cancel{8}^{\,2}\cdot 64}{3}-64\right) = \underline{\underline{\frac{64}{3}}}.$$

3) i) $A=\{x\in\mathbb{R}: \log_{\frac12}(3+x)+\log_{\frac12}x \geq -2\}$

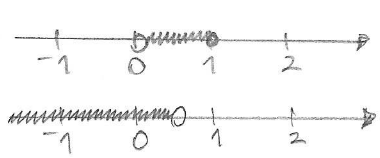
$$\begin{cases} 3+x>0 \\ x>0 \\ (3+x)x\leq 4 \end{cases} \Leftrightarrow \begin{cases} x>-3 \\ x>0 \\ x^2+3x-4\leq 0 \end{cases} \Leftrightarrow \begin{cases} x>-3 \\ x>0 \\ (x+4)(x-1)\leq 0 \end{cases}$$

$$\begin{cases} x>-3 \\ x>0 \\ -4\leq x\leq 1 \end{cases} \Rightarrow \underline{\underline{A=]0,1]}}.$$

$B=\left\{x\in\mathbb{R}: \frac{3x-|x+1|}{2^x}<0\right\}$. Poiché $2^x>0\ \forall x\in\mathbb{R}$, basta determinare gli $x\in\mathbb{R}$ t.c. $3x-|x+1|<0$. Si ha

$$\begin{cases} x+1\geq 0 \\ 3x-x-1<0 \end{cases} \text{ o } \begin{cases} x+1<0 \\ 3x+x+1<0 \end{cases} \Leftrightarrow$$

$$\begin{cases} x\geq -1 \\ 2x<1 \end{cases} \text{ o } \begin{cases} x<-1 \\ 4x<-1 \end{cases} \Rightarrow \underline{\underline{B=]-\infty,\tfrac12[}}.$$



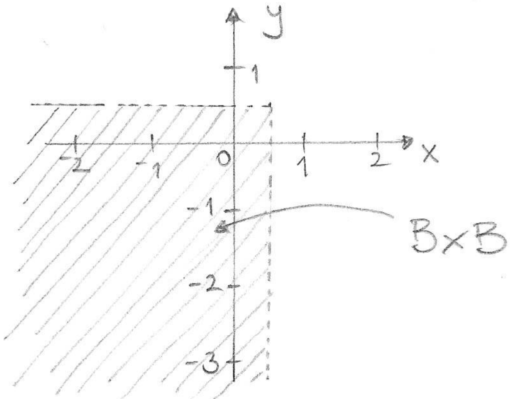
//// = A

//// = B.

$\max A=1$ $\nexists \min A$; $\nexists$ono $\max B$, $\min B$. □

ii) $A\cap B=\underline{\underline{]0,\tfrac12[}}$; è un intervallo. □

iii)



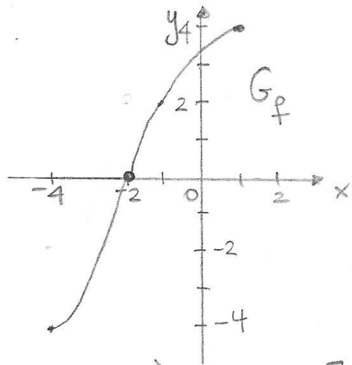
4) $f : [-4,1] \to \mathbb{R}$

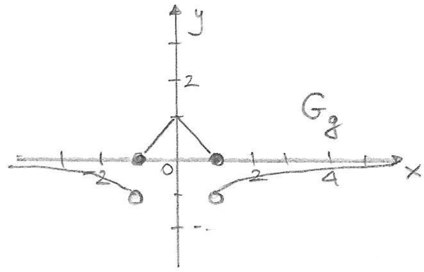
$$f(x) = \begin{cases} (x+4)^2 - 2 & \text{se } -4 \le x < -2 \\ 2\log_2(x+3) & \text{se } -2 \le x \le 1 \end{cases}$$

$g : \mathbb{R} \to \mathbb{R}$

$$g(x) = \begin{cases} -|x| + 1 & \text{se } x \in [-1,1] \\ -\frac{1}{x^2} & \text{se } x \in \mathbb{R} \setminus [-1,1] \end{cases}$$

i)





ii) $f([-4,1]) = \underline{\underline{[-4,4]}}$.

$f^{-1}(2) = x \iff f(x) = 2$; risulta $\underline{\underline{x = -1}}$

$f^{-1}(4) = x \iff f(x) = 4$; risulta $\underline{\underline{x = 1}}$.

iii) $(f \circ g)(x)$ per $x \in [-1,1]$: osserviamo che

$g([-1,1]) = [0,1]$; ora $f(x) = 2\log_2(x+3)$ per $x \in [-2,1]$,

quindi

$$(f \circ g)(x) \underset{x \in [-1,1]}{=} f(g(x)) = 2\log_2(g(x)+3) = 2\log_2(-|x|+1+3) =$$
$$= \underline{\underline{2\log_2(-|x|+4)}}.$$

iv) g non soddisfa le ipotesi del teorema di Weierstrass su $[-3,3]$,

non essendo continua in $x = -1$ e $x = 1$

(per es. $\lim_{x \to -1^-} \left(-\frac{1}{x^2}\right) = -1 \neq \lim_{x \to -1^+} (-|x|+1) = 0$).

v) $\min_{\mathbb{R}} |g(x)| = 0$ $\quad x \in \{-1,1\}$ pt. di minimo

$\max_{\mathbb{R}} |g(x)| = 1$ $\quad x = 0$ pt. di massimo.

5) $f(x) = 5x^2 - 2x^5$ è – la funzione studiata in FILA (A) Es. 5 i)

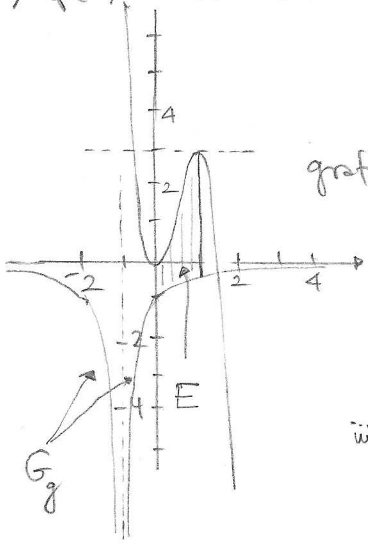


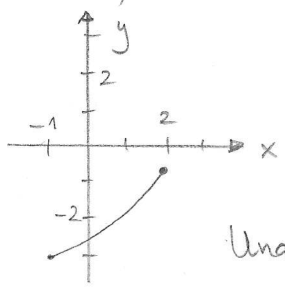
grafico qualitativo di f

ii) $\underline{y = 3}$.

iii) $\text{area}(E) = \int_0^1 \left(5x^2 - 2x^5 + \frac{1}{(x+1)^2}\right) dx = \frac{11}{6}$.

6) $f : [-1, 2] \to \mathbb{R}$, $F(x) = \int_{-1}^{x} f(t)\,dt$;

dal teorema fond. del calcolo integrale abbiamo che $f(x) < 0 \Rightarrow F$ decrescente, mentre $f'(x) > 0 \Rightarrow F$ convessa!



Una possibile f!