Università di Trento – Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
SECONDA PROVA INTERMEDIA DI ANALISI MATEMATICA
a.a. 2014–2015 — Rovereto, 16 dicembre 2014

FILA A

1) Risolvete in $\mathbb{R}$

• $x(2-|3-x|)\geq 0$:

$$\begin{cases} 3-x\geq 0 \\ x(2-(3-x))\geq 0 \end{cases} \quad \text{o} \quad \begin{cases} 3-x<0 \\ x(2+(3-x))\geq 0 \end{cases}$$

$$\iff \begin{cases} x\leq 3 \\ x(x-1)\geq 0 \end{cases} \quad \text{o} \quad \begin{cases} x>3 \\ x(-x+5)\geq 0 \end{cases}$$

$$\iff \begin{cases} x\leq 3 \\ x\leq 0 \ \text{ o } \ x\geq 1 \end{cases} \quad \text{o} \quad \begin{cases} x>3 \\ 0\leq x\leq 5 \end{cases}$$

$$\iff \quad x\in \,]-\infty,0]\cup[1,3] \qquad x\in\,]3,5]$$

$$\Rightarrow \quad \underline{\underline{S=\,]-\infty,0]\cup[1,5].}}$$

•• $\dfrac{3^x\,81^x}{3^{-|x|}}\leq 3$:

$$\iff 3^x\cdot 3^{4x}\cdot 3^{|x|}\leq 3^1 \iff 3^{5x+|x|}\leq 3^1$$

$$\underset{3^x\uparrow}{\iff} \quad 5x+|x|\leq 1 \iff \begin{cases} x\geq 0 \\ 6x\leq 1 \end{cases} \quad \text{o} \quad \begin{cases} x<0 \\ 4x\leq 1 \end{cases}$$

$$\Rightarrow \quad \underline{\underline{S=\,]-\infty,1/6].}}$$

••• $\log_2(x+1)+\log_{\frac{1}{2}}(x-x^2)\geq 1 \iff \log_2(x+1)-\log_2(x-x^2)\geq \log_2 2$

$$\iff \log_2(x+1)\geq \log_2 2+\log_2(x-x^2)$$

$$\Rightarrow \begin{cases} x+1>0 \\ x-x^2>0 \\ \log_2(x+1)\geq \log_2\left[2(x-x^2)\right] \end{cases} \iff \begin{cases} x>-1 \\ 0<x<1 \\ x+1\geq 2(x-x^2) \end{cases}$$

$$\iff \begin{cases} 0<x<1 \\ 2x^2-x+1\geq 0 \end{cases} \quad \Rightarrow \quad \underline{\underline{S=\,]0,1[.}}$$

2) i) $$\int_{-1}^{3}|x^2-4|\,dx = \int_{-1}^{2}(-x^2+4)\,dx+\int_{2}^{3}(x^2-4)\,dx=\left[-\frac{x^3}{3}+4x\right]_{-1}^{2}+\left[\frac{x^3}{3}-4x\right]_{2}^{3}$$

$x^2-4\geq 0 \iff x\leq -2 \ \text{o}\ x\geq 2$

$$=\left(-\frac{8}{3}+8\right)-\left(\frac{1}{3}-4\right)+(9-12)-\left(\frac{8}{3}-8\right)=$$

$$=-\frac{17}{3}+17=\frac{2}{3}\cdot 17=\underline{\underline{\frac{34}{3}}}.$$

$$\int_1^2 \frac{x^3-\sqrt{x}}{2x}\,dx = \frac{1}{2}\int_1^2 \left(x^2 - x^{-\frac{1}{2}}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^2 = \frac{1}{2}\left[\left(\frac{8}{3}-2\sqrt{2}\right)-\left(\frac{1}{3}-2\right)\right]$$

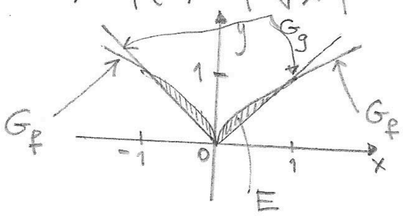
$$= \frac{1}{2}\left[\frac{7}{3} - 2\sqrt{2} + 2\right] = \underline{\underline{\frac{13}{6} - \sqrt{2}}}.$$

$$\int_0^1 \frac{3x}{x^2+1}\,dx = \frac{3}{2}\int_0^1 \frac{2x}{x^2+1}\,dx = \frac{3}{2}\left[\log(1+x^2)\right]_0^1 = \frac{3}{2}\left[\log 2 - \cancel{\log 1}\right]$$

$$= \underline{\underline{\frac{3}{2}\log 2}}.$$

$\square$

ii) $f(x) = \left|\sqrt[3]{x}\right| \qquad g(x) = |x|$



$$\text{area } E = 2\int_0^1 \left(f(x)-g(x)\right)dx = 2\int_0^1 \left[\sqrt[3]{x} - x\right]dx$$

$$= 2\left[\frac{x^{4/3}}{4/3} - \frac{x^2}{2}\right]_0^1 = 2\left[\frac{3}{4}-\frac{1}{2}\right] = 2\cdot\frac{1}{4} = \underline{\underline{\frac{1}{2}}}.$$

$\square$

iii) $$\sum_{n=0}^{5} f(n!) = f(1) + f(1) + f(2) + f(6) + f(24) + f(120)$$

$$= 2 + 2 + 2^2 + \log 6 + \log 24 + \log 120.$$

$$= 8 + \log(3\cdot 2) + \log(4\cdot 3\cdot 2) + \log(5\cdot 4\cdot 3\cdot 2) =$$

$$= 8 + 3\log 2 + 3\log 3 + 2\log 4 + \log 5.$$

$\blacksquare$

3) $f: \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{3x}-1 & \text{se } x \le 0 \\ \frac{2}{x}+1 & \text{se } 0 < x < 2 \\ -x^2+4x & \text{se } x \ge 2 \end{cases}$$

$$-x^2+4x \searrow -(x^2-4x) = -(x-2)^2+4$$

i) $$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(e^{3x}-1\right) = \underline{\underline{-1}};$$

$\underline{y=-1}$ asintoto orizz. per f, per $x \to -\infty$.

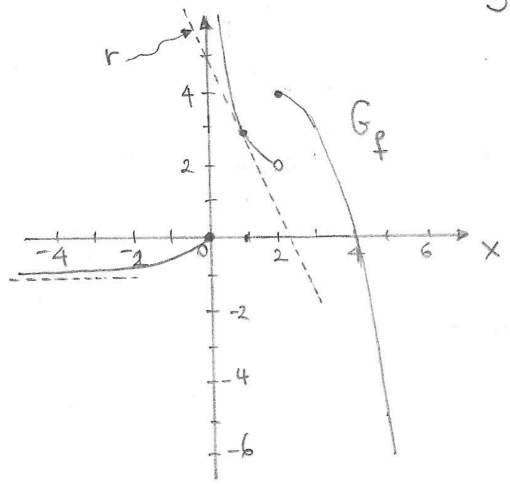
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{2}{x}+1\right) = \underline{\underline{+\infty}};$$

$\underline{x=0}$ asintoto verticale (destro) per f.

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(\frac{2}{x}+1\right) = \underline{\underline{2}};$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(-x^2-4x\right) = \underline{\underline{-\infty}}.$$

ii)



iii) $\underline{\underline{x=0}}$ $\quad f(0)=0 \neq \lim_{x\to 0^+} f(x)$;

$\underline{\underline{x=2}}$ $\quad f(2)=4 \neq \lim_{x\to 2^-} f(x)$.

☐

iv) f non soddisfa le ipotesi del teorema di Weierstrass su $[1,4]$, poiché non è continua in $[1,4]$.

f ha comunque massimo e minimo su $[1,4]$ (ricordate che il teorema di Weierstrass vi dà cond. sufficienti affinché il massimo e il minimo esistano, ma non necessari).

Si ha $\max_{[1,4]} f = \underline{\underline{4}}$; $\quad \min_{[1,4]} f = \underline{\underline{0}}$.

☐

v) Oss. che $f(x) = \frac{2}{x}+1$ nell'intervallo $]0,2[$, e $f'(x) = -\frac{2}{x^2}$

Quindi l'eq. della retta tgr al grafico di f in $x_0 = 1$ è data da

$$y = f(1) + f'(1)(x-1)$$
$$= 3 - 2(x-1) \quad , \quad \text{ossia} \quad \underline{\underline{y = -2x+5}}.$$

4a) i) F è crescente su $[1,3]$ e decrescente su $[3,5]$.

☐

ii) $x=1$ pt. di minimo locale ; $F(1)=0$.

$x=3$ pt. di massimo locale di F $\left(F(3)=\int_1^3 f(t)dt = \text{area}\right.$ del sottografico di f su $[1,3]$ = area E_1.

$x=5$ pt. di minimo locale di F.

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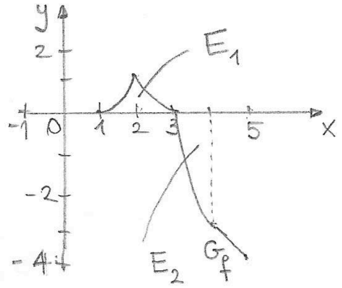
b) F è una funzione continua (essendo derivabile per il TFC(*), poiché f continua su $[1,5]$). Inoltre $F(3) = \text{area } E_1 > 0$



mentre $F(4) = \text{area } E_1 - \text{area } E_2$
$$< \frac{\cancel{2}\cdot 1}{\cancel{2}} - \frac{1\cdot 3}{2} < 0.$$

Quindi $\exists\, x_0 \in\,]3,4[$ t.c. $F(x_0)=0$ (per il teorema di esistenza degli zeri).

(*) TFC = teorema fondamentale del Calcolo Integrale

5) i) $\boxed{f(x) = \dfrac{x^2-2}{x+1}}$

- $\mathrm{dom} f = \mathbb{R} \setminus \{-1\} =]-\infty, -1[\cup]-1, +\infty[$.

- segno di f

x^2-2	$+$	$+$	$-\sqrt{2}$	$-$	$-$	$\sqrt{2}$	$+$	$+$
$x+1$	$-$	$-$	$-$	-1	$+$	$+$	$+$	$+$
$f(x)$	$-$	$-$	$-\sqrt{2}$	$+$	-1	$-$	$\sqrt{2}$	$+$

- comportamento agli estremi del dominio

$\lim_{x \to -\infty} f(x) = -\infty$ — $y = x - 1$ asintoto obliquo per f, per $x \to -\infty$;

$\lim_{x \to -1^-} f(x) = +\infty$ — $x = -1$ asintoto verticale (sinistro) per f;

$\lim_{x \to -1^+} f(x) = -\infty$ — "

$\lim_{x \to +\infty} f(x) = +\infty$ — $y = x - 1$ asintoto obliquo per f, per $x \to +\infty$.

- f è continua su $\mathrm{dom} f$ (somma e rapporto di funz. continue).

- $\mathrm{dom} f' = \mathrm{dom} f$

$$f'(x) = \frac{2x(x+1) - (x^2-2)}{(x+1)^2} = \frac{x^2+2x+2}{(x+1)^2} > 0$$

$f \nearrow$ su $]-\infty, -1[$, $\nearrow$ su $]-1, +\infty[$.

- $\mathrm{dom} f'' = \mathrm{dom} f$

$$f''(x) = \frac{(2x+2)(x+1)^{\cancel{2}} - (x^2+2x+2)\,2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}} = \frac{\cancel{2x^2} + \cancel{4x} + 2 - \cancel{2x^2} - \cancel{4x} - 4}{(x+1)^3}$$

$$= \frac{-2}{(x+1)^3}$$

f''	$+$	-1	$-$	$-$
f	$\cup$		$\cap$	

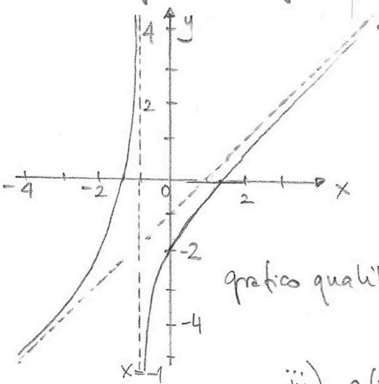


grafico qualitativo di f

ii) $\mathrm{dom}\, g(x) = \{x \in \mathrm{dom} f : f(x) > 0\}$

$]-\sqrt{2}, -1[\cup]\sqrt{2}, +\infty[$

iii) $g(x) = 0$ per $x \in \mathrm{dom}\, g \iff f(x) = 1$ per $x \in \mathrm{dom}\, g$. Ora

$f(x) = 1 \iff \dfrac{x^2-2}{x+1} = 1 \iff x^2 - 2 = x + 1 \iff x^2 - x - 3 = 0$

$$x_{1/2} = \frac{1 \pm \sqrt{1+12}}{2} = \frac{1 \pm \sqrt{13}}{2}.$$

6) $D_{4,2} = \dfrac{4!}{2!} = \dfrac{4 \cdot 3 \cdot \cancel{2!}}{\cancel{2!}} = 12$.

FILA B

1) Risolvere in $\mathbb{R}$

• $x(2-|4-x|)<0$: $\begin{cases} 4-x \geq 0 \\ x(2-4+x)<0 \end{cases}$ o $\begin{cases} 4-x<0 \\ x(2+4-x)<0 \end{cases}$

$\Leftrightarrow \begin{cases} x \leq 4 \\ x(x-2)<0 \end{cases}$ o $\begin{cases} x>4 \\ x(6-x)<0 \end{cases}$

$\Leftrightarrow \begin{cases} x \leq 4 \\ 0<x<2 \end{cases}$ o $\begin{cases} x>4 \\ x<0 \text{ o } x>6 \end{cases}$

$\Rightarrow \underline{S =]0,2[\cup]6,+\infty[}.$

•• $\dfrac{3^x \, 81^x}{3^{|x|}} \geq 3$: $\Leftrightarrow 3^x \cdot 3^{4x} \cdot 3^{-|x|} \geq 3^1 \Leftrightarrow 3^{5x-|x|} \geq 3^1$

$\underset{3^x\nearrow}{\Leftrightarrow} 5x-|x| \geq 1 \Leftrightarrow \begin{cases} x \geq 0 \\ 4x \geq 1 \end{cases}$ o $\begin{cases} x<0 \\ 6x \geq 1 \end{cases}$

$\Rightarrow \underline{S = [\tfrac{1}{4}, +\infty[}.$

••• $\log_3(x-1) + \log_{1/3}(x+x^2) \leq 1 \Leftrightarrow \log_3(x-1) - \log_3(x+x^2) \leq \log_3 3$

$\Leftrightarrow \log_3(x-1) \leq \log_3 3 + \log_3(x+x^2)$

$\Rightarrow \begin{cases} x-1>0 \\ x+x^2>0 \\ x-1 \leq 3(x+x^2) \end{cases} \Leftrightarrow \begin{cases} x>1 \\ x<-1 \text{ o } x>0 \\ 3x^2+2x+1 \geq 0 \end{cases}$

$\Rightarrow \underline{S =]1,+\infty[}$

2) i) $\displaystyle\int_{-3}^{2} |x^2-4|\,dx = \int_{-3}^{-2} (x^2-4)\,dx + \int_{-2}^{2} (-x^2+4)\,dx = \left[\frac{x^3}{3}-4x\right]_{-3}^{-2} + \left[-\frac{x^3}{3}+4x\right]_{-2}^{2} =$

$x^2-4 \geq 0 \Leftrightarrow x \leq -2 \text{ o } x \geq 2$

$= \left[\left(-\frac{8}{3}+8\right)-(-9+12)\right] + \left[\left(-\frac{8}{3}+8\right)-\left(+\frac{8}{3}-8\right)\right] =$

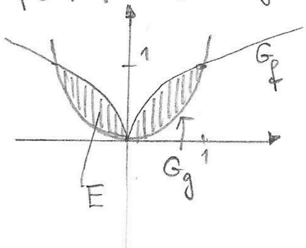
$= -8+3\cdot 8-3 = \underline{13}.$

$\displaystyle\int_1^2 \frac{x^4-\sqrt{x}}{2x}\,dx = \frac{1}{2}\int_1^2 \left[x^3 - x^{-1/2}\right] dx = \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^{1/2}}{1/2}\right]_1^2 = \frac{1}{2}\left[(4-2\sqrt{2})-\left(\frac{1}{4}-2\right)\right]$

$= \frac{1}{2}\left[-\frac{1}{4}-2\sqrt{2}+6\right] = \underline{\frac{23}{8}-\sqrt{2}}.$

$$\int_0^1 x e^{x^2-1}\,dx = \frac{1}{2}\int_0^1 2x e^{x^2-1}\,dx = \frac{1}{2}\left[e^{x^2-1}\right]_0^1 = \frac{1}{2}\left[e^0 - e^{-1}\right] = \underline{\underline{\frac{1}{2}\left(1-\frac{1}{e}\right)}} \quad \square$$

ii) $f(x) = \left|\sqrt[3]{x}\right| \qquad g(x) = x^2$



$$\text{area } E = 2\int_0^1 \left(f(x) - g(x)\right)dx = 2\int_0^1 \left(\sqrt[3]{x} - x^2\right)dx =$$

$$= 2\left[\frac{x^{4/3}}{4/3} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{3}{4} - \frac{1}{3}\right] = \cancel{2}\cdot\frac{5}{\cancel{12}_6} = \underline{\underline{\frac{5}{6}}}.$$

iii) $$\sum_{n=0}^{6} f(n!) = f(1) + f(1) + f(2) + f(6) + f(24) + f(120) + f(720)$$

$$= e + e + e^2 + \log 6 + \log 24 + \log 120 + \log 720.$$

$$= 2e + e^2 + \log(3\cdot 2) + \log(4\cdot 3\cdot 2) + \log(5\cdot 4\cdot 3\cdot 2) + \log(6\cdot 5\cdot 4\cdot 3\cdot 2)$$

$$= 2e + e^2 + 4\log 2 + 4\log 3 + 3\log 4 + 2\log 5 + \log 6.$$

3) $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} -e^{3x} + 1 & \text{se } x \le 0 \\ \dfrac{2}{x} + 1 & \text{se } 0 < x < 2 \\ -x^2 + 4x - 3 & \text{se } x \ge 2. \end{cases}$$

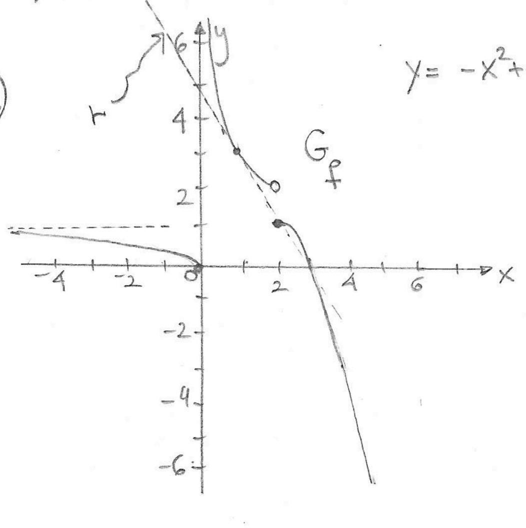
i) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(-e^{3x}+1\right) = \underline{\underline{1}}$; $\qquad$ $\underline{y = 1}$ asintoto orizzontale per f, per $x \to +\infty$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(\frac{2}{x} + 1\right) = \underline{\underline{+\infty}}$; $\qquad$ $\underline{x = 0 \text{ asintoto verticale (destro) per } f}$.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(\frac{2}{x} + 1\right) = \underline{\underline{2}}$;

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \left(-x^2+4x-3\right) = \underline{\underline{-\infty}}$.

ii)



$$y = -x^2 + 4x - 3 = -\left(x^2 - 4x\right) - 3 = -\left(x-2\right)^2 + 1$$

iii) $\underline{\underline{x = 0}} \qquad f(0) = 0 \ne \lim_{x\to 0^+} f(x)$;

$\underline{\underline{x = 2}} \qquad f(2) = 1 \ne \lim_{x\to 2^-} f(x)$.

$\square$

iv) f non soddisfa le ipotesi del teorema di Weierstrass su $[1,4]$, poiché f non è continua su $[1,4]$.

f ha comunque massimo e minimo su $[1,4]$ (il teorema di Weierstrass ci dà cond. sufficienti affinché il massimo e il minimo esistano, ma non necessarie).

Si ha $\max\limits_{[1,4]} f = f(1) = \underline{\underline{3}}$; $\min\limits_{[1,4]} f = f(4) = \underline{\underline{-3}}$. □

v) $\underline{\underline{y = -2x+5}}$ (vedi Fila A es. 3)v)). ■

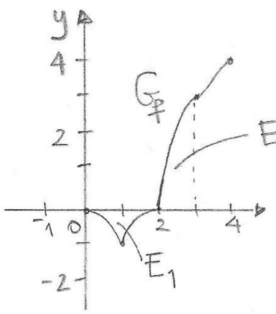
4) a) i) F è decrescente su $[0,2]$ e crescente su $[2,4]$. □

ii) $x=0$ pt. di massimo locale; $F(0)=0$.

$x=2$ pt. di minimo locale di F

$\left(F(2) = \int_0^2 f(t)\,dt = -\text{area } E_1\right.$.

$x=4$ pt. di massimo locale di F. □



b) F è una funzione continua (essendo derivabile per il TFC(*), poiché f continua su $[0,4]$). Inoltre

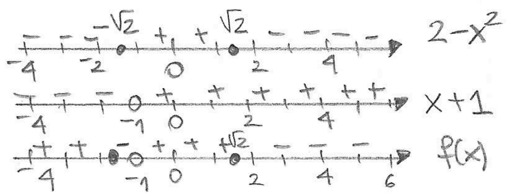
$F(2) = -\text{area } E_1 < 0$, mentre $F(3) = -\text{area } E_1 + \text{area } E_2 > -\frac{\cancel{2}\cdot 1}{\cancel{2}} + \frac{1\cdot 3}{2} > 0$.

Quindi $\exists\, x_0 \in \,]2,3[$ t.c. $F(x_0)=0$ (teorema di esistenza degli zeri). ■

5) i) $\boxed{f(x) = \dfrac{2-x^2}{x+1}}$

• $\operatorname{dom} f = \mathbb{R}\setminus\{-1\} = \,]-\infty,-1[\, \cup \,]-1,+\infty[$.

• segno di f



(*) TFC = teorema fondamentale del calcolo integrale

- comportamento agli estremi del dominio

$\lim_{x\to-\infty} f(x) = +\infty$ $\qquad$ $\underline{y = -x+1 \text{ asintoto obliquo per } f, \text{ per } x\to-\infty;}$

$\lim_{x\to-1^-} f(x) = -\infty$ $\qquad$ $\underline{x = -1 \text{ asintoto verticale (sinistro) per } f;}$

$\lim_{x\to-1^+} f(x) = +\infty$ $\qquad$ "

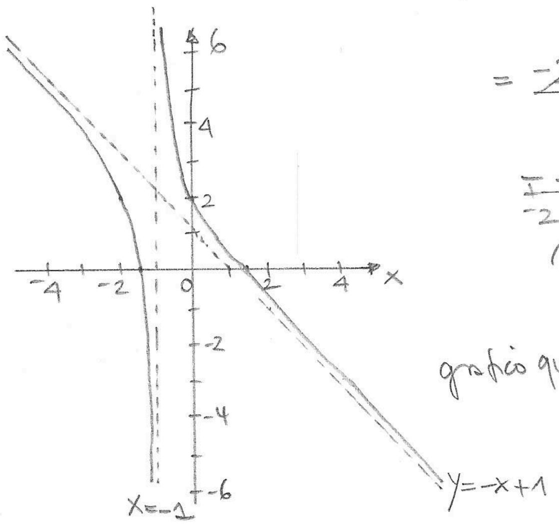
$\lim_{x\to+\infty} f(x) = -\infty$ $\qquad$ $\underline{y = -x+1 \text{ asintoto obliquo per } f, \text{ per } x\to+\infty.}$

- f è continua in dom f (somma e rapporto di funz. continue).
- dom f' = dom f $\qquad f'(x) = \dfrac{-2x(x+1)-(2-x^2)}{(x+1)^2} = \dfrac{-x^2-2x-2}{(x+1)^2} < 0 \Rightarrow$

$f\downarrow$ in $]-\infty,-1[$
$f\downarrow$ in $]-1,+\infty[$

- dom f'' = dom f $\qquad f''(x) = \dfrac{(-2x-2)(x+1)^{\cancel{2}} - (-x^2-2x-2)2\cancel{(x+1)}}{(x+1)^{\cancel{4}3}} =$

$$= \frac{\cancel{-2x^2}\cancel{-4x}-2+\cancel{2x^2}+\cancel{4x}+4}{(x+1)^3} = \frac{2}{(x+1)^3}$$



$- \; - \; 0 \; + \; + \; + \; + \; + \; + \rightarrow f''$
$\quad -2 \quad -1 \quad 0 \quad 1 \quad 2$
$\cap \qquad\qquad \cup \qquad f$

grafico qualitativo di f

ii) dom $g(x) = \{x \in \text{dom} f : f(x) > 0\}$

$\underline{]-\infty,-\sqrt{2}[\,\cup\,]-1,\sqrt{2}[.}$ $\square$

iii) $g(x)=0$ per $x\in$ dom $g \iff f(x)=1$ per $x \in$ dom g. Ora

$$f(x)=1 \iff \frac{2-x^2}{x+1} = 1 \iff 2-x^2 = x+1 \iff$$

$$\iff x^2+x-1=0 \qquad x_{1/2} = \frac{-1\pm\sqrt{1+4}}{2} = \underline{\underline{\frac{-1\pm\sqrt{5}}{2}}}$$

6) $D_{6,2} = \dfrac{6!}{4!} = \dfrac{6\cdot5\cdot\cancel{4!}}{\cancel{4!}} = \underline{\underline{30}}.$