E2

1) Determinate modulo, coniugato, reciproco dei seg. numeri complessi:

$z = 1-2i$ $\quad |z| = \sqrt{1+4} = \sqrt{5}$ $\quad \bar{z} = 1+2i$ $\quad \frac{1}{z} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1}{5}+\frac{2}{5}i.$

$z = -3+i$ $\quad |z| = \sqrt{9+1} = \sqrt{10}$ $\quad \bar{z} = -3-i$ $\quad \frac{1}{z} = \frac{-3-i}{(-3+i)(-3-i)} = -\frac{3}{10}-\frac{1}{10}i.$

$z = \sqrt{3}+i$ $\quad |z| = \sqrt{3+1} = 2$ $\quad \bar{z} = \sqrt{3}-i$ $\quad \frac{1}{z} = \frac{\sqrt{3}-i}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{\sqrt{3}}{4}-\frac{1}{4}i.$

$z = -2-\frac{1}{2}i$ $\quad |z| = \sqrt{4+\frac{1}{4}} = \frac{\sqrt{17}}{2}$ $\quad \bar{z} = -2+\frac{1}{2}i$ $\quad \frac{1}{z} = \frac{-2+\frac{1}{2}i}{\frac{17}{4}} = -\frac{8}{17}+\frac{2}{17}i.$

$z = 8i$ $\quad |z| = 8$ $\quad \bar{z} = -8i$ $\quad \frac{1}{z} = \frac{-8i}{64} = -\frac{1}{8}i.$

$z = \frac{1}{1-i} - \frac{2i}{1-i} = \frac{1-2i}{1-i}\cdot\frac{1+i}{1+i} = \frac{3-i}{2} = \frac{3}{2}-\frac{1}{2}i$; quindi

$|z| = \sqrt{\frac{9}{4}+\frac{1}{4}} = \frac{\sqrt{10}}{2}$ $\quad \bar{z} = \frac{3}{2}+\frac{1}{2}i$ $\quad \frac{1}{z} = \frac{\frac{3}{2}+\frac{1}{2}i}{\frac{10}{4}} =$

$= \frac{3}{5}+\frac{1}{5}i.$ ▨

2) $-i$ $\quad z = \cos\left(-\frac{\pi}{2}\right)+i\,\mathrm{sen}\left(-\frac{\pi}{2}\right)$ $\quad z = e^{-i\pi/2}$.

$2-2i$ $\quad z = 2\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right) =$

$= 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\,\mathrm{sen}\left(-\frac{\pi}{4}\right)\right)$ $\quad z = 2\sqrt{2}\,e^{-i\pi/4}$.

$3+\sqrt{3}i$ $\quad z = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) =$

$= 2\sqrt{3}\left(\cos\frac{\pi}{6}+i\,\mathrm{sen}\frac{\pi}{6}\right)$ $\quad z = 2\sqrt{3}\,e^{i\pi/6}$.

$-\sqrt{3}+3i$ $\quad z = 2\sqrt{3}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$

$= 2\sqrt{3}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$ $\quad z = 2\sqrt{3}\,e^{i\,2\pi/3}$.

$2i$ $\quad z = 2\left(\cos\frac{\pi}{2}+i\,\mathrm{sen}\frac{\pi}{2}\right)$ $\quad z = 2e^{i\pi/2}$.

-4 $\quad z = 4\left(\cos\pi+i\,\mathrm{sen}\,\pi\right)$ $\quad z = 4e^{i\pi}$ ▨

3) i) $\boxed{w = 3+\sqrt{3}i} = 2\sqrt{3}\left(\cos\frac{\pi}{6}+i\,\mathrm{sen}\frac{\pi}{6}\right)$

radici quadrate: $z_0 = \sqrt{2\sqrt{3}}\left(\cos\frac{\pi}{12}+i\,\mathrm{sen}\frac{\pi}{12}\right)$ $\quad z_1 = \sqrt{2\sqrt{3}}\left(\cos\left(\frac{\pi}{12}+\pi\right)+i\,\mathrm{sen}\left(\frac{\pi}{12}+\pi\right)\right).$

radici cubiche: $z_0 = \sqrt[3]{2\sqrt{3}}\left(\cos\frac{\pi}{18} + i \sin\frac{\pi}{18}\right)$ $z_1 = \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{18}+\frac{2\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{18}+\frac{2\pi}{3}\right)\right)$

$z_2 = \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{18}+\frac{4\pi}{3}\right) + i \operatorname{sen}\left(\frac{\pi}{18}+\frac{4\pi}{3}\right)\right)$. □

$\boxed{w=4} = 4(\cos 0 + i \operatorname{sen} 0)$

radici quadrate: $z_0 = 2(\cos 0 + i \operatorname{sen} 0) = 2$ $z_1 = 2(\cos\pi + i \operatorname{sen}\pi) = -2$.

radici cubiche: $z_0 = \sqrt[3]{4}$ $z_1 = \sqrt[3]{4}\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right)$ $z_2 = \sqrt[3]{4}\left(\cos\left(\frac{4\pi}{3}\right) + i \operatorname{sen}\frac{4\pi}{3}\right)$

■

ii) $\boxed{w = 2+2i} = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right)$

radici quarte: $z_0 = \sqrt[4]{2\sqrt{2}}\left(\cos\frac{\pi}{16} + i \operatorname{sen}\frac{\pi}{16}\right)$ $z_1 = \sqrt[4]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{16}+\frac{\pi}{2}\right) + i \operatorname{sen}\left(\frac{\pi}{16}+\frac{\pi}{2}\right)\right)$

$z_2 = \sqrt[4]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{16}+\pi\right) + i \operatorname{sen}\left(\frac{\pi}{16}+\pi\right)\right)$ $z_3 = \sqrt[4]{2\sqrt{2}}\left(\cos\left(\frac{\pi}{16}+\frac{3}{2}\pi\right) + i \operatorname{sen}\left(\frac{\pi}{16}+\frac{3\pi}{2}\right)\right)$

□

$\boxed{w = -\sqrt{3}+3i} = 2\sqrt{3}\left(\cos\frac{2\pi}{3} + i \operatorname{sen}\frac{2\pi}{3}\right)$

radici quinte: $z_0 = \sqrt[5]{2\sqrt{3}}\left(\cos\frac{2\pi}{15} + i \operatorname{sen}\frac{2\pi}{15}\right)$ $z_1 = \sqrt[5]{2\sqrt{3}}\left(\cos\left(\frac{2\pi}{15}+\frac{2\pi}{5}\right) + i \operatorname{sen}\left(\frac{2\pi}{15}+\frac{2\pi}{5}\right)\right)$

$z_2 = \sqrt[5]{2\sqrt{3}}\left(\cos\left(\frac{2\pi}{15}+\frac{4\pi}{5}\right) + i \operatorname{sen}\left(\frac{2\pi}{15}+\frac{4\pi}{5}\right)\right)$

$z_3 = \sqrt[5]{2\sqrt{3}}\left(\cos\left(\frac{2\pi}{15}+\frac{6\pi}{5}\right) + i \operatorname{sen}\left(\frac{2\pi}{15}+\frac{6\pi}{5}\right)\right)$

$z_4 = \sqrt[5]{2\sqrt{3}}\left(\cos\left(\frac{2\pi}{15}+\frac{8\pi}{5}\right) + i \operatorname{sen}\left(\frac{2\pi}{15}+\frac{8\pi}{5}\right)\right)$

■

4) $\dfrac{iz + 2|z|^2}{\bar{z}+2}$ per $\boxed{z = 2+3i}$:

$$\frac{i(2+3i)+2(4+9)}{2-3i+2} = \frac{2i-3+26}{4-3i} = \frac{23+2i}{4-3i} =$$

$$= \frac{(23+2i)(4+3i)}{25} = \frac{(92-6)+i(69+8)}{25} =$$

$$= \underline{\underline{\frac{86}{25} + \frac{77}{25}i}}$$

per $\boxed{z = 4i}$:

$$\frac{i\,4i + 2\cdot 16}{-4i+2} = \frac{-4+32}{2-4i}\cdot\frac{2+4i}{2+4i} = \frac{56+112i}{20}$$

$$= \underline{\underline{\frac{14}{5} + \frac{28}{5}i}}.$$

□

$\left|\frac{z-1}{\bar{z}+1}\right|$ per $\boxed{z=2+3i}$:

$$\left|\frac{2+3i-1}{2-3i+1}\right| = \left|\frac{1+3i}{3-3i}\right| = \frac{\sqrt{10}}{\sqrt{18}} = \sqrt{\frac{\cancel{2}\cdot 5}{9\cdot\cancel{2}}} = \frac{\sqrt{5}}{3}.$$

per $\boxed{z=4i}$;

$$\left|\frac{4i-1}{-4i+1}\right| = 1 .$$

□

$\operatorname{Im}\left(i z\bar{z} + \frac{|z|^2}{z}\right) = \operatorname{Im}\left(i|z|^2 + \bar{z}\right)$ per $\boxed{z=2+3i}$;

$$= \operatorname{Im}\left(i(4+9) + (2-3i)\right) = \operatorname{Im}(2+10i) = 10 ;$$

per $\boxed{z=4i}$;

$$\operatorname{Im}(i\,16 - 4i) = \operatorname{Im}(12i) = 12 .$$

□

$\operatorname{Re}\left(\frac{|z|^2 - 2\bar{z}}{iz}\right)$ per $\boxed{z=2+3i}$:

$$\operatorname{Re}\left(\frac{13-2(2-3i)}{-3+2i}\right) = \operatorname{Re}\left(\frac{(9+6i)(-3-2i)}{13}\right) =$$

$$= \operatorname{Re}\left(\frac{(-27+12)-36i}{13}\right) = -\frac{15}{13}.$$

per $\boxed{z=4i}$:

$$\operatorname{Re}\left(\frac{16+8i}{i\,4i}\right) = \operatorname{Re}\left(\frac{16+8i}{-4}\right) = -4$$

■

5) i) $\frac{4}{1+i} = \frac{4(1-i)}{2} = 2-2i$;

$$\frac{2+i}{3-i} + i - 3 = \frac{(2+i)(3+i)}{10} + i - 3 = \frac{(6-1)+5i}{10} + i - 3 = \frac{1}{2} - 3 + \frac{1}{2}i + i$$

$$= -\frac{5}{2} + \frac{3}{2}i ;$$

$$\frac{-3i}{1+i} = \frac{-3i}{1+i}\cdot\frac{1-i}{1-i} = \frac{-3-3i}{2} = -\frac{3}{2} - \frac{3}{2}i ;$$

$$\frac{i^2-3i}{1+2i} = \frac{-1-3i}{1+2i}\cdot\frac{1-2i}{1-2i} = \frac{(-1-6)-i}{1+4} = \underline{\underline{-\frac{7}{5}-\frac{1}{5}i}}$$

□

ii) $$\left(\frac{1+i}{1-i}\right)^2 = \frac{[(1+i)(1+i)]^2}{4} = \frac{(2i)^2}{4} = \underline{\underline{-1}}\ ;$$

$$\frac{(8-i)+(6+i)}{2+2i} = \frac{14}{2+2i} = \frac{7}{1+i}\cdot\frac{1-i}{1-i} = \underline{\underline{\frac{7}{2}-\frac{7}{2}i}}\ ;$$

$$\frac{3i}{|2-i|^2} = \underline{\underline{\frac{3}{5}i}}\ ;$$

$$\frac{4+2i}{i} = \frac{4+2i}{i}\cdot\frac{-i}{-i} = \underline{\underline{2-4i}}$$

■

6) a) $\frac{z}{|z|}+z-2=0$: nota $z=0$ non è soluzione. Quindi l'eq. data è equiv. $\frac{|z|^2}{\cancel{|z|}}+|z|^2-2\bar{z}=0$

$\bar{z} = \frac{|z|+|z|^2}{2}$. Allora $\bar{z}$ è un numero reale, e anche z lo è. Poniamo $z=x$ e si ha

$\frac{x}{|x|}+x-2=0$. Se $x>0$, allora $1+x-2=0$ e si ha $x=1$ e quindi $\underline{z=1}$;

Se $x<0$, allora $-1+x-2=0 \iff x=3$, ma questa soluzione non è accettabile. □

$\underline{\underline{S=\{1\}}}$.

$z-3i=\bar{z}+1$: poniamo $z=x+iy$ e si ha

$$x+iy-3i = x-iy+1 \iff \begin{cases} x=x+1 & \text{impossibile} \\ y-3=-y \end{cases}$$

$\Rightarrow \underline{\underline{S=\emptyset}}$ □

b) $z\bar{z}-|z|=0 \iff |z|^2-|z|=0 \iff |z|(|z|-1)=0$

$$\underline{\underline{S=\{0,\ \cos\vartheta+i\sin\vartheta\}_{\vartheta\in[0,2\pi[}}}$$

ossia $z=0$ e tutti i pt. sulla circonferenza di centro O e raggio 1. □

$z^2+z+(3+2i\sqrt{3})/4=0$

E2

$$z_{1/2} = \frac{-1 \pm \sqrt{1-(3+2i\sqrt{3})}}{2} = \frac{-1 \pm \sqrt{-2-2i\sqrt{3}}}{2}$$

Consideriamo $w = -2-2i\sqrt{3} = 4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 4\left(\cos\frac{4\pi}{3} + i \operatorname{sen}\frac{4\pi}{3}\right)$

$$\sqrt{w} = 2\left(\cos\frac{2\pi}{3} + i \operatorname{sen}\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$$

$$z_{1/2} = \frac{-1 \pm (-1+i\sqrt{3})}{2} \Rightarrow \quad \underline{z_1 = -1 + i\frac{\sqrt{3}}{2}} \quad \underline{z_2 = -i\frac{\sqrt{3}}{2}}$$

c) $z^2 - 2iz - (3+i\sqrt{3})/2 = 0$

$$z_{1/2} = \frac{2i \pm \sqrt{-4+2(3+i\sqrt{3})}}{2} = \frac{2i \pm \sqrt{2+2i\sqrt{3}}}{2}$$

Consideriamo $w = 2+2i\sqrt{3} = 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 4\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)$

$$\sqrt{w} = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$z_{1/2} = \frac{2i \pm (\sqrt{3}+i)}{2} \Rightarrow \underline{z_1 = \frac{\sqrt{3}+3i}{2}} \quad \underline{z_2 = \frac{-\sqrt{3}+i}{2}} \quad \square$$

$z(2\bar{z} - \operatorname{Re} z) = 4 - i$, Poniamo $z = x+iy$ e si ha

$$(x+iy)(2x-2iy-x) = 4-i$$

$$(x+iy)(x-2iy) = 4-i$$

$$x^2+2y^2-xyi = 4-i \iff \begin{cases} x^2+2y^2 = 4 \\ xy = 1 \end{cases} \iff y = 1/x \text{ poiché } x=0 \text{ non è ammissibile}$$

parabole equilatere; ellisse di centro $(0,0)$ e semiassi $a=2$ $b=\sqrt{2}$

$$\iff \begin{cases} x^2 + 2\cdot\frac{1}{x^2} = 4 \\ xy = 1 \end{cases} \iff \begin{cases} x^4 - 4x^2 + 2 = 0 \\ y = 1/x \end{cases}$$

$$x^2 = t \qquad t^2 - 4t + 2 = 0 \qquad t_{1/2} = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$$\underline{z_1 = \sqrt{2+\sqrt{2}} + i\frac{1}{\sqrt{2+\sqrt{2}}}} \qquad \underline{z_2 = -\sqrt{2+\sqrt{2}} - i\frac{1}{\sqrt{2+\sqrt{2}}}}$$

$$\underline{z_3 = \sqrt{2-\sqrt{2}} + i\frac{1}{\sqrt{2-\sqrt{2}}}} \qquad \underline{z_4 = -\sqrt{2-\sqrt{2}} - i\frac{1}{\sqrt{2-\sqrt{2}}}} . \quad \blacksquare$$

d) $z^4 = 3z \iff z(z^3-3) = 0 \iff z = 0 \text{ o } z^3 - 3 = 0$

$$z_0 = \sqrt[3]{3} \quad z_1 = \sqrt[3]{3}\left(\cos\frac{2\pi}{3} + i \operatorname{sen}\frac{2\pi}{3}\right) \quad z_2 = \sqrt[3]{3}\left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}\right)$$

e quindi le soluzioni sono $S = \{0, z_0, z_1, z_2\}$. $\square$

$z^2-3i\bar z=0$: $z=x+iy$ e si ha $(x+iy)^2-3i(x-iy)=0$

$\iff x^2-y^2+2xyi-3ix-3y=0$

$\iff \begin{cases} x^2-y^2-3y=0 \\ 2xy-3x=0 \end{cases} \iff \begin{cases} x^2-y^2-3y=0 \\ x(2y-3)=0 \end{cases}$

$\begin{cases} x=0 \\ y^2+3y=0 \end{cases}$ o $\begin{cases} y=3/2 \\ x^2-9/4-\frac{9}{2}=0 \end{cases} \iff \begin{cases} y=3/2 \\ x^2=\frac{27}{4} \end{cases}$

$\underline{\underline{z_0=0}}$ $\underline{\underline{z_1=-3i}}$ $\underline{\underline{z_2=\frac{3\sqrt3}{2}+\frac{3}{2}i}}$ $\underline{\underline{z_3=-\frac{3\sqrt3}{2}+\frac{3}{2}i}}$.

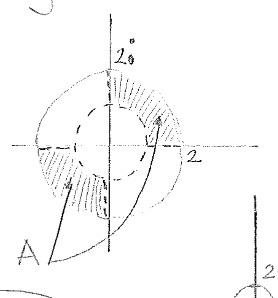
7) $z=-2\sqrt3+2i=4\left(-\frac{\sqrt3}{2}+\frac{1}{2}i\right)=4\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$\underline{\underline{z^4}}=4^4\left(\cos\frac{5\pi}{\cancel{6}_3}\cdot\cancel{4}^2+i\sin\frac{5\pi}{\cancel{6}_3}\cdot\cancel{4}^2\right)=\underline{\underline{16^2\left(\cos\frac{10\pi}{3}+i\sin\frac{10\pi}{3}\right)}}$

$\underline{\underline{z^{10}}}=4^{10}\left(\cos\frac{5\pi}{\cancel{6}_3}\cdot\cancel{10}^5+i\sin\frac{5\pi}{\cancel{6}_3}\cdot\cancel{10}^5\right)=\underline{\underline{4^{10}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}}$.

8) $A=\{z\in\mathbb{C} : z^4=16,\ |\mathrm{Im}z|>|\mathrm{Re}z|\}=\underline{\underline{\{2i,-2i\}}}$

9) i) $A=\{z\in\mathbb{C} : 1<|z|\le 2,\ (\mathrm{Re}z)(\mathrm{Im}z)>0\}$

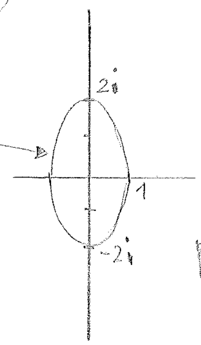


ii) $B=\left\{z\in\mathbb{C} : |z|^2-\frac{3(\mathrm{Im}z)^2}{4}=\log_4 4\right\}$

$=\left\{z\in\mathbb{C} : |z|^2-3\frac{(\mathrm{Im}z)^2}{4}=1\right\}$

$=\left\{\underset{x+iy}{\underset{\parallel}{z}}\in\mathbb{C} : x^2+y^2-\frac{3y^2}{\cancel{4}}=1\right\}$

$x^2+\frac{y^2}{4}=1$



10) i) $\begin{cases} z+iw\bar z=-i \\ w-iz\bar w=i \end{cases} \iff \begin{matrix} z+iw\bar z=-i \\ \underline{\underline{\bar w+i\bar z w=-i}} \end{matrix}$

sottraendo $\Rightarrow z=\bar w$ o equiv. $\bar z=w$. Inserendo nella 1ª eq. si ottiene allora $z+i(\bar z)^2=-i$. Procediamo ponendo $z=x+iy$ e si ha $x+iy+i(x-iy)^2=-i$, ossia

$x+iy+i(x^2-y^2-2xyi)=-i$.

E2

si ha $\begin{cases} x + 2xy = 0 \\ y + x^2 - y^2 = -1 \end{cases} \iff \begin{cases} x(1+2y) = 0 \\ y + x^2 - y^2 = -1 \end{cases}$

Se $x = 0$ dalla 2ª eq. si ha $y^2 - y - 1 = 0$ $\quad y_{1/2} = \frac{1 \pm \sqrt{5}}{2}$

Se $x \neq 0$, si ha $y = -\frac{1}{2}$, e dalla 2ª eq. si ottiene

$$-\frac{1}{2} + x^2 - \frac{1}{4} = -1 \qquad x^2 = -\frac{1}{4} \quad \text{impossibile!}$$

Allora si ha che $\left(z_1 = \frac{1-\sqrt{5}}{2} i \; , \; w_1 = -\frac{1+\sqrt{5}}{2} i\right)$,

$$\left(z_2 = \frac{1+\sqrt{5}}{2} i \; , \; w_2 = -\frac{1+\sqrt{5}}{2} i\right)$$

sono le due coppie che risolvono il sistema. ▨

ii) $\begin{cases} \bar{z} w = i \\ |w|^2 z + 1 = 0 \end{cases} \iff \begin{cases} \bar{z} w = i \\ w \bar{w} z + 1 = 0 \end{cases} \Rightarrow \begin{cases} z \bar{w} = -i \\ w(-i) + 1 = 0 \end{cases}$

$\Rightarrow \begin{cases} z \bar{w} = -i \\ w = \frac{1}{i} \cdot \frac{-i}{-i} = -i \end{cases} \iff \begin{cases} z i = -i \\ w = -i \end{cases} \iff \begin{cases} z = -1 \\ w = -i \end{cases}$ è l'unica coppia soluzione

▨