E6

1) a) $\lim_{x\to 0} \frac{x-\sin x\cos x}{x^3}$ $\left[\frac{0}{0}\right]$

Oss. $\frac{[x-\sin x\cos x]'}{(x^3)'} = \frac{1-\cos^2 x+\sin^2 x}{3x^2} = \frac{\sin^2 x+\sin^2 x}{3x^2}$

$= \frac{2}{3}\frac{\sin^2 x}{x^2} \xrightarrow{x\to 0} \frac{2}{3}$

Appl. il [teor. di H] si conclude che $\lim_{x\to 0}\frac{x-\sin x\cos x}{x^3} = \boxed{\frac{2}{3}}$. □

b) $\lim_{x\to+\infty} \frac{\log(1+\sqrt{x})}{\log x}$ $\left[\frac{\infty}{\infty}\right]$

Oss. $\frac{[\log(1+\sqrt{x})]'}{[\log x]'} = \frac{\frac{1}{1+\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}{1/x} = \frac{x}{2x+2\sqrt{x}} \xrightarrow{x\to+\infty} \frac{1}{2}$

Appl. il [teor. di H] si conclude che $\lim_{x\to+\infty}\frac{\log(1+\sqrt{x})}{\log x} = \boxed{\frac{1}{2}}$. □

c) $\lim_{x\to 0^+} \frac{\log x}{\log(\sin x)}$ $\left[\frac{\infty}{\infty}\right]$

Oss. $\frac{[\log x]'}{[\log(\sin x)]'} = \frac{\frac{1}{x}}{\frac{1}{\sin x}\cdot\cos x} = \frac{\sin x}{x\cos x} \xrightarrow[x\to 0]{} 1$

Appl. il [teor. di H] si conclude che $\lim_{x\to 0^+}\frac{\log x}{\log(\sin x)} = \boxed{1}$. □

d) $\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{x-\sin x}{x\sin x}\right)$ $\left[\frac{0}{0}\right]$

Oss. che $\frac{[x-\sin x]'}{[x\sin x]'} = \frac{1-\cos x}{\sin x+x\cos x}$ $\left[\frac{0}{0}\right]$ e

$\frac{[1-\cos x]'}{[\sin x+x\cos x]'} = \frac{+\sin x}{\cos x+\cos x-x\sin x} \xrightarrow[x\to 0^+]{} 0$

Appl. due volte [teor. di H] si può concludere che

$\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \boxed{0}$. □

e) $\lim_{x\to 0^+} (\arcsin x)\log x$

Oss. $(\arcsin x)\log x = \frac{\log x}{\frac{1}{\arcsin x}}$ $\left[\frac{\infty}{\infty}\right]$

E6

Oss. $\dfrac{[\log x]'}{\left(\dfrac{1}{\arcsin x}\right)'} = \dfrac{\dfrac{1}{x}}{\dfrac{-1}{(\arcsin x)^2}\cdot\dfrac{1}{\sqrt{1-x^2}}} = -\dfrac{\sqrt{1-x^2}}{x}\cdot(\arcsin x)^2$

$$= -\underbrace{x\sqrt{1-x^2}}_{\downarrow\, 0}\cdot\underbrace{\left(\frac{\arcsin x}{x}\right)^2}_{\downarrow\, x\to 0 \; 1} \xrightarrow{x\to 0^+} 0 .$$

Appl. il [teor. di H] si conclude che $\lim_{x\to 0^+}(\arcsin x)\log x = \boxed{0}$.

□

f) $\lim_{x\to 1}\dfrac{\log x}{\tan(\pi x)}$ $\quad \left[\dfrac{0}{0}\right]$

Oss. $\dfrac{[\log x]'}{[\tan(\pi x)]'} = \dfrac{\dfrac{1}{x}}{\dfrac{1}{\cos^2(\pi x)}\cdot\pi} = \dfrac{1}{\pi}\,\dfrac{\cos^2\pi x}{x} \xrightarrow{x\to 1} \dfrac{1}{\pi}$.

Appl. il [teor. di H] si conclude che $\lim_{x\to 1}\dfrac{\log x}{\tan(\pi x)} = \boxed{\dfrac{1}{\pi}}$.

□

g) $\lim_{x\to +\infty} x\left(\operatorname{arctg} x - \dfrac{\pi}{2}\right)$ $\quad \infty\cdot 0$

Oss. $x\left(\operatorname{arctg} x - \pi/2\right) = \dfrac{\operatorname{arctg} x - \pi/2}{1/x}$ $\quad \left[\dfrac{0}{0}\right]$

Allo. $\dfrac{\left[\operatorname{arctg} x - \frac{\pi}{2}\right]'}{\left(1/x\right)'} = \dfrac{\dfrac{1}{1+x^2}}{-1/x^2} = -\dfrac{x^2}{1+x^2} \to -1$

Appl. il [teor. di H] si conclude che

$\lim_{x\to +\infty} x\left(\operatorname{arctg} x - \dfrac{\pi}{2}\right) = \boxed{-1}$.

■

2) i) a) $\lim_{x\to 0^+} \frac{\cos\sqrt{x}-\cos x}{2x}$ $\left[\frac{0}{0}\right]$:

Oss. che $\frac{[\cos\sqrt{x}-\cos x]'}{(2x)'} = \frac{-\underbrace{\frac{1}{2\sqrt{x}}\sin\sqrt{x}}_{\xrightarrow[x\to 0^+]{} 1}+\sin x}{2} \xrightarrow[x\to 0^+]{} -\frac{1}{4}$.

Applicando il teorema di de L'Hôpital (scriveremo (H) da ora in poi) possiamo asserire che

$$\lim_{x\to 0^+} \frac{\cos\sqrt{x}-\cos x}{2x} = \boxed{-\frac{1}{4}}.$$

□

b) $\lim_{x\to+\infty} \left(x\cos\frac{1}{\sqrt{x}} - x^2\sin\frac{1}{x}\right)$:

Oss. $\lim_{x\to+\infty} \frac{\cos\frac{1}{\sqrt{x}}}{1/x} - \frac{\sin\frac{1}{x}}{1/x^2} \underset{y=\frac{1}{x}}{\overset{\uparrow}{=}} \lim_{y\to 0^+}\left[\frac{\cos\sqrt{y}}{y} - \frac{\sin y}{y^2}\right]$ $\quad y\to 0^+ \Leftrightarrow x\to+\infty$

$$= \lim_{y\to 0^+}\left[\frac{y\cos\sqrt{y}-\sin y}{y^2}\right] \quad \left[\frac{0}{0}\right].$$

Notiamo che

$$\frac{[y\cos\sqrt{y}-\sin y]'}{(y^2)'} = \frac{\cos\sqrt{y} - \frac{y}{2\sqrt{y}}(\sin\sqrt{y}) - \cos y}{2y}$$

$$= \frac{\cos\sqrt{y}-\cos y}{2y} - \frac{1}{4}\frac{\sin\sqrt{y}}{\sqrt{y}} \xrightarrow[a)]{y\to 0^+} -\frac{1}{4}-\frac{1}{4} = -\frac{1}{2}$$

Applicando (H) si ha

$$\lim_{x\to+\infty}\left(x\cos\frac{1}{\sqrt{x}} - x^2\sin\frac{1}{x}\right) = \boxed{-\frac{1}{2}}$$

■

ii) a) $\lim_{x\to+\infty} \frac{e^{\left(\sin\frac{1}{x}\right)^{-1}}}{e^x} = \lim_{x\to+\infty} e^{\frac{1}{\sin 1/x}-x} = \lim_{y\to 0^+} e^{\frac{1}{\sin y}-\frac{1}{y}}$.

Oss. $\frac{1}{\sin y} - \frac{1}{y} = \frac{y-\sin y}{y\sin y}$ $\left[\frac{0}{0}\right]$, e

$\frac{[y-\sin y]'}{[y\sin y]'} = \frac{1-\cos y}{\sin y + y\cos y}$ $\left[\frac{0}{0}\right]$ e

$\frac{[1-\cos y]'}{[\sin y + y\cos y]'} = \frac{\sin y}{\cos y + \cos y - y\sin y} \xrightarrow[y\to 0^+]{} 0$.

e quindi, grazie a (H) si ha $\lim_{y\to 0^+} e^{\frac{1}{\sin y}-\frac{1}{y}} = e^0 = 1$,

e quindi
$$\lim_{x\to+\infty} \frac{e^{(\sin\frac{1}{x})^{-1}}}{e^x} = \boxed{1}.$$
$\square$

b) $$\lim_{x\to+\infty} \frac{e^{(\sin\frac{1}{x})^{-2}}}{e^{x^2}} = \lim_{x\to+\infty} e^{\frac{1}{(\sin\frac{1}{x})^2}-x^2} = \lim_{y\to 0^+} e^{\frac{1}{(\sin y)^2}-\frac{1}{y^2}}$$

Oss. $$\frac{1}{(\sin y)^2}-\frac{1}{y^2} = \frac{y^2-\sin^2 y}{y^2(\sin y)^2} = \frac{\cancel{y^2}-\cancel{y^2}+\frac{1}{3}y^4+o(y^4)}{y^4+o(y^4)} \xrightarrow[y\to 0^+]{} \frac{1}{3}$$

quindi $\lim_{y\to 0^+} e^{\frac{1}{(\sin y)^2}-\frac{1}{y^2}} = e^{1/3}$ e quindi
$$\lim_{x\to+\infty} \frac{e^{(\sin\frac{1}{x})^{-2}}}{e^{x^2}} = \boxed{e^{\frac{1}{3}}}.$$
$\blacksquare$

3) a) $$\lim_{x\to 0} \frac{\log^2(1+\sin x)}{x\,\mathrm{tg}\,x} = \boxed{1}.$$

Infatti: $$\frac{\log^2(1+\sin x)}{x\,\mathrm{tg}\,x} = \frac{\log^2(1+x+o(x))}{x(x+o(x))} = \frac{x^2+o(x^2)}{x^2+o(x^2)} \xrightarrow[x\to 0]{} 1$$

Si poteva anche usare i limiti notevoli $$\underbrace{\frac{\log^2(1+\sin x)}{(\sin x)^2}}_{\to 1} \cdot \underbrace{\frac{(\sin x)^2}{x\cdot \cancel{\sin x}}}_{\to 1} \cdot \underbrace{\cos x}_{\to 1} \to 1.$$
$\square$

b) $$\lim_{x\to 0} \frac{x-\log(1+x)}{1-\cos x} = \boxed{1}.$$

Infatti $$\frac{x-\log(1+x)}{1-\cos x} = \frac{\cancel{x}-\cancel{x}+\frac{x^2}{2}+o(x^2)}{\frac{x^2}{2}+o(x^2)} \xrightarrow[x\to 0]{} 1.$$
$\square$

c) $$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)} = \boxed{e^{-1/6}}.$$

Infatti $$\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right) = \frac{1}{x^2}\log\left(\frac{x-\frac{x^3}{6}+o(x^3)}{x}\right) = \frac{1}{x^2}\log\left(1-\frac{x^2}{6}+o(x^2)\right)$$
$$= \frac{1}{x^2}\left(-\frac{x^2}{6}+o(x^2)\right) = -\frac{1}{6}+o(1) \xrightarrow[x\to 0]{} -\frac{1}{6}.$$
$\square$

d) $\lim\limits_{x\to0}\left(\frac{\sin x}{x}\right)^{1/x} = \lim\limits_{x\to0} e^{\frac{1}{x}\log\left(\frac{\operatorname{sen} x}{x}\right)} = e^0 = \boxed{1}$.

<u>Infatti</u> $\frac{1}{x}\log\left(\frac{\operatorname{sen} x}{x}\right) = \frac{1}{x}\log\left(\frac{x-\frac{x^3}{6}+o(x^3)}{x}\right) = \frac{1}{x}\log\left(1-\frac{x^2}{6}+o(x^2)\right)$

$= \frac{1}{x}\left(-\frac{x^2}{6}+o(x^2)\right) = -\frac{x}{6}+o(x) \xrightarrow[x\to0]{} 0$ □

e) $\lim\limits_{x\to0}\left(\frac{1}{x\operatorname{tg} x}-\frac{1}{x^2}\right) = \lim\limits_{x\to0}\frac{x-\operatorname{tg} x}{x^2\operatorname{tg} x} = \boxed{-\frac{1}{3}}$.

<u>Infatti</u> $\frac{x-\operatorname{tg} x}{x^2\operatorname{tg} x} = \frac{\cancel{x}-\cancel{x}-\frac{x^3}{3}+o(x^3)}{x^3+o(x^3)} \longrightarrow -\frac{1}{3}$.

$\boxed{\operatorname{tg} x = x+\frac{x^3}{3}+o(x^3)}$ □

f) $\lim\limits_{x\to0}\frac{1-\cos x^5}{\left(\sin x - x\cos(x/\sqrt{3})\right)^2} = \boxed{\frac{(270)^2}{2}}$

<u>Infatti</u> $\frac{1-\cos x^5}{\left(\sin x - x\cos(x/\sqrt{3})\right)^2} = \frac{\cancel{1}-\cancel{1}+\frac{x^{10}}{2}+o(x^{10})}{\left(\cancel{x}-\cancel{\frac{x^3}{3!}}+\frac{x^5}{5!}+o(x^5)-\cancel{x}+\cancel{\frac{x\cdot x^2}{2\cdot3}}-\frac{x\cdot x^4}{4!\,9}\right)^2}$

$= \frac{x^{10}/2+o(x^{10})}{\left(x^5\left(\frac{1}{5!}-\frac{1}{4!\,9}\right)+o(x^5)\right)^2} = \frac{x^{10}/2+o(x^{10})}{x^{10}\left(\frac{1}{5!}-\frac{1}{4!\,9}\right)^2+o(x^{10})} \longrightarrow \frac{(270)^2}{2}$. □

g) $\lim\limits_{x\to0}\frac{1}{x^2}\left(\frac{\sin x}{x}-\frac{x}{\sin x}\right) = \boxed{-\frac{1}{3}}$.

<u>Infatti</u> $\frac{1}{x^2}\cdot\left[\frac{\sin^2 x - x^2}{x\sin x}\right] = \frac{\left(x-\frac{x^3}{6}+o(x^3)\right)^2-x^2}{x^3(x+o(x))} =$

$= \frac{\cancel{x^2}-\frac{2}{6}x^4+o(x^4)-\cancel{x^2}}{x^4+o(x^4)} \xrightarrow[x\to0]{} -\frac{1}{3}$. □

h) $\lim\limits_{x\to0}\frac{x^2(\sin x^2-\sin^2 x)}{\sin x^2-\operatorname{tg} x^2} = \boxed{-\frac{2}{3}}$.

$\frac{x^2(\sin x^2-\sin^2 x)}{\sin x^2-\operatorname{tg} x^2} = \frac{x^2\left(x^2-\frac{x^6}{6}+o(x^6)-\left(x-\frac{x^3}{3!}+o(x^3)\right)^2\right)}{\cancel{x^2}-\frac{x^6}{6}+o(x^6)-\cancel{x^2}-\frac{x^6}{3}+o(x^6)}$

$= \frac{x^2\left(\cancel{x^2}-\frac{x^6}{6}+o(x^6)-\cancel{x^2}+\frac{\cancel{2}\cdot x^4}{\cancel{6}\,3}+o(x^4)\right)}{-\frac{x^6}{2}+o(x^6)} = \frac{x^6/3+o(x^6)}{-\frac{x^6}{2}+o(x^6)}$

$\xrightarrow[x\to0]{} -\frac{2}{3}$. □

i) $\lim_{x\to 0^+} \frac{x-\sin^2\sqrt{x}-\sin^2 x}{x^2} = \boxed{-\frac{2}{3}}$.

<u>Infatti</u>, $\frac{x-\sin^2\sqrt{x}-\sin^2 x}{x^2} = \frac{x-\left(\sqrt{x}-\frac{\sqrt{x}^3}{3!}+o(\sqrt{x}^3)\right)^2-\left(x-\frac{x^3}{3!}+o(x^3)\right)^2}{x^2}$

$= \frac{\cancel{x}-\cancel{x}+2\frac{x^2}{3!}+o(x^2)-x^2+o(x^2)}{x^2} = \frac{-\frac{2}{3}x^2+o(x^2)}{x^2}$ $\square$

$= -\frac{2}{3}+o(1) \xrightarrow[x\to 0^+]{} -\frac{2}{3}$.

j) $\lim_{x\to 0} \frac{\left[(1-x)^{-1}+e^x\right]^2-4e^{2x}-2x^2}{x^3} = \boxed{\frac{16}{3}}$.

<u>Infatti</u>, $\frac{\left[(1-x)^{-1}+e^x\right]^2-4e^{2x}-2x^2}{x^3} = \frac{\left[1+x+x^2+x^3+o(x^3)+1+x+\frac{x^2}{2}+\frac{x^3}{6}+o(x^3)\right]^2}{x^3}$

$\frac{-4\left(1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{3!}+o(x^3)\right)-2x^2}{x^3}$

$= \frac{\left[2+2x+\frac{3}{2}x^2+\frac{7}{6}x^3+o(x^3)\right]^2-4-8x-8x^2-\frac{32}{6}x^3+o(x^3)-2x^2}{x^3}$

$= \frac{\cancel{4}+\cancel{8x}+\cancel{10x^2}+\frac{32x^3}{3}-\cancel{4}-\cancel{8x}-\cancel{10x^2}-\frac{32}{6}x^3+o(x^3)}{x^3}$

$= \frac{\frac{\cancel{32}\,16}{\cancel{6}\,3}x^3+o(x^3)}{x^3} = \frac{16}{3}+o(1) \xrightarrow[x\to 0]{} \frac{16}{3}$. $\square$

k) $\lim_{x\to 0^+} \frac{\operatorname{sen}^2 x+3(x-\operatorname{sen}^2\sqrt{x})}{x^2} = \boxed{2}$.

<u>Infatti</u>, $\operatorname{sen}^2 x = \left(x-\frac{x^3}{6}+o(x^3)\right)^2 = x^2-o(x^2)$

$3(x-\operatorname{sen}^2\sqrt{x}) = 3\left(x-\left(\sqrt{x}-\frac{(\sqrt{x})^3}{3!}+o(\sqrt{x}^3)\right)^2\right) =$

$= 3\left(\cancel{x}-\cancel{x}+\frac{x^2}{3}+o(x^2)\right) = x^2+o(x^2)$

$\frac{\operatorname{sen}^2 x+3(x-\sin^2\sqrt{x})}{x^2} = \frac{2x^2+o(x^2)}{x^2} = 2+o(1) \xrightarrow[x\to 0]{} 2$.

$\square$

l) $\displaystyle\lim_{x\to 0} \frac{x\cos x - \sin x}{1+x^2-e^{x^2}+\sin^3 x} = \boxed{-\frac{1}{3}}$.

Infatti, $\displaystyle\frac{x\cos x - \sin x}{1+x^2-e^{x^2}+\sin^3 x} = \frac{x\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^4)\right)-\left(x-\frac{x^3}{6}+\frac{x^5}{5!}+o(x^5)\right)}{\cancel{1}+\cancel{x^2}-\cancel{1}-\cancel{x^2}-\frac{(x^2)^2}{2}+o(x^4)+x^3+o(x^3)}$

$$= \frac{\cancel{x}-\frac{x^3}{2}+o(x^3)-\cancel{x}+\frac{x^3}{6}+o(x^3)}{x^3+o(x^3)} = \frac{-\frac{\cancel{2}^1}{\cancel{6}_3}x^3+o(x^3)}{x^3+o(x^3)}$$

$$= \frac{-\frac{1}{3}+o(1)}{1+o(1)} \xrightarrow[x\to 0]{} -\frac{1}{3}. \qquad \square$$

m) $\displaystyle\lim_{x\to 0} \frac{(e^x-1)^3-\log^2(1-x)}{1-\cos^2 x} = \boxed{-1}$.

Infatti, $(e^x-1)^3 = \left(x+\frac{x^2}{2}+\frac{x^3}{3!}+o(x^3)\right)^3$

$\log^2(1-x) = \left(-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+o(x^4)\right)^2$

$1-\cos^2 x = 1-\left(1-\frac{x^2}{2}+o(x^2)\right)^2 =$

$= \cancel{1}-\cancel{1}+x^2+o(x^2)$

$$\frac{(e^x-1)^3-\log^2(1-x)}{1-\cos^2 x} = \frac{x^3+o(x^3)-x^2+o(x^2)}{x^2+o(x^2)} = \frac{-x^2+o(x^2)}{x^2+o(x^2)}$$

$$= \frac{-1+o(1)}{1+o(1)} \xrightarrow[x\to 0]{} -1. \qquad \square$$

n) $\displaystyle\lim_{x\to 0} \frac{e^{x^2}-1-\log(1+x\,\mathrm{arctg}\,x)}{\sqrt{1+2x^4}-1} = \boxed{\frac{4}{3}}$.

Infatti $\sqrt{1+2x^4} = (1+2x^4)^{1/2} = 1+\frac{1}{\cancel{2}}(\cancel{2}x^4)+o(x^4)$

$\sqrt{1+2x^4}-1 = x^4+o(x^4)$

$e^{x^2}-1-\log\left(1+x\left(x-\frac{x^3}{3}+o(x^3)\right)\right) = \cancel{1}+x^2+\frac{x^4}{2}+o(x^4)-\cancel{1}-\log\left(1+x^2-\frac{x^4}{3}+o(x^4)\right)$

$= x^2+\frac{x^4}{2}+o(x^4)-\left(x^2-\frac{x^4}{3}+o(x^4)\right)+\frac{\left(x^2-\frac{x^4}{3}+o(x^4)\right)^2}{2}+o(x^4)$

$= \frac{x^4}{2}+\frac{x^4}{3}+\frac{x^4}{2}+o(x^4) = 4\frac{x^4}{3}+o(x^4)$; allora

$$\frac{e^{x^2}-1-\log(1+x\,\mathrm{arctg}\,x)}{\sqrt{1+2x^4}-1} = \frac{\frac{4}{3}x^4+o(x^4)}{x^4+o(x^4)} \xrightarrow[x\to 0]{} \frac{4}{3}. \qquad \blacksquare$$