Soluzioni del foglio d'esercizi recupero I

Es. 1.1 Risolvere le seg. disequazioni:

1) $$\sqrt{x^2-9} < x+1 \iff \begin{cases} x^2-9 \geq 0 \\ x+1 \geq 0 \\ x^2-9 < (x+1)^2 \end{cases} \iff \begin{cases} x \leq -3 \text{ o } x \geq 3 \\ x \geq -1 \\ \cancel{x^2}-9 < \cancel{x^2}+2x+1 \end{cases}$$

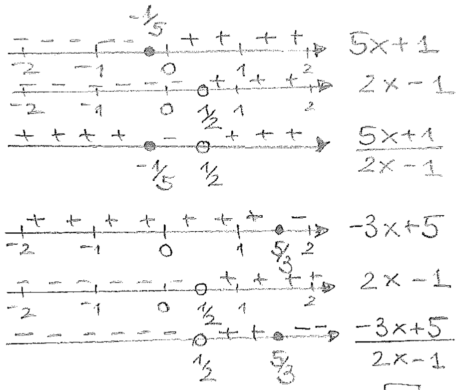
$$\iff \begin{cases} x \geq 3 \\ x > -5 \end{cases} \Rightarrow \underline{\underline{S=[3,+\infty[}}.$$ ☐

2) $$\left|\frac{x+3}{2x-1}\right| < 2 \iff -2 < \frac{x+3}{2x-1} < 2 \iff \begin{cases} \frac{x+3}{2x-1}+2>0 \\ \frac{x+3}{2x-1}-2<0 \end{cases}$$

$$\iff \begin{cases} \frac{5x+1}{2x-1} > 0 \\ \frac{-3x+5}{2x-1} < 0 \end{cases} \iff$$

$$\iff \begin{cases} x \in \left]-\infty,-\frac{1}{5}\right[\cup \left]\frac{1}{2},+\infty\right[\\ x \in \left]-\infty,\frac{1}{2}\right[\cup \left]\frac{5}{3},+\infty\right[\end{cases}$$

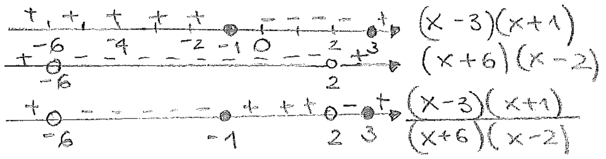
$$\Rightarrow \underline{\underline{S = \left]-\infty,-\frac{1}{5}\right[\cup \left]\frac{5}{3},+\infty\right[}}$$



☐

3) $$\frac{|x^2-2x|-3}{x^2+4x-12} \geq 0 \iff \begin{cases} x^2-2x \geq 0 \\ \frac{x^2-2x-3}{x^2+4x-12} \geq 0 \end{cases} \text{ o } \begin{cases} x^2-2x<0 \\ \frac{-x^2+2x-3}{x^2+4x-12} \geq 0 \end{cases}$$

$$\iff \begin{cases} x \leq 0 \text{ o } x \geq 2 \\ \frac{(x-3)(x+1)}{(x+6)(x-2)} \geq 0 \end{cases} \text{ o } \begin{cases} 0<x<2 \\ \frac{x^2-2x+3}{(x+6)(x-2)} \leq 0 \end{cases}$$



Nota: $x^2-2x+3>0 \ \forall x \in \mathbb{R}$,
quindi $\frac{x^2-2x+3}{(x+6)(x-2)} \leq 0 \iff (x+6)(x-2)<0$

$$\iff x \in \left]-\infty,-6\right[\cup [-1,0] \cup [3,+\infty[\qquad x \in \left]0,2\right[$$

$$\Rightarrow \underline{\underline{S = \left]-\infty,-6\right[\cup [-1,2[\cup [3,+\infty[}}.$$

▨

<u>Es.1.2</u> Studiare la limitatezza dei seguenti insiemi numerici, determinando per ognuno di essi sup, inf, max e min.

1) $A=\left\{a_n=\frac{1}{3^{n^2-3n+1}} : n\geq 0\right\}$. Si ha subito che $a_n>0 \ \forall n\geq 0$.

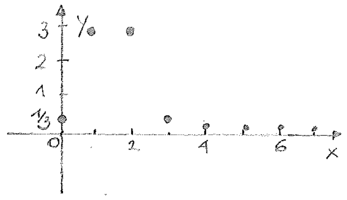
Sia $b_n=n^2-3n+1$. Risulta $b_{n+1}\geq b_n \iff$

$$(n+1)^2-3(n+1)+\cancel{1} \geq n^2-3n+\cancel{1}$$

$$\iff \cancel{n^2}+2n+1-\cancel{3n}-3\geq \cancel{n^2}-\cancel{3n}$$

$$\iff n\geq 1 .$$

Poiché 3^x è una funzione crescente, risulta $3^{n^2-3n+1}\nearrow$ per $n\geq 1$, e quindi $a_n=\frac{1}{3^{n^2-3n+1}}\searrow$ per $n\geq 1$.

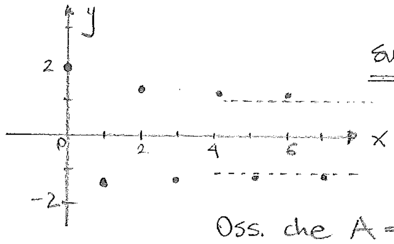


Risulta quindi $\underline{\inf A}=\lim_{n\to+\infty}a_n=\underline{0}$ (nota: $a_0=\frac{1}{3}>0$)
mentre $\underline{\sup A}=\underline{\max A}=a_1$ (ma anche a_2) $=\frac{1}{3^{-1}}=\underline{3}$.

Oss. che $A=$ immagine della funzione $a_n:\mathbb{N}\to\mathbb{R}$ $\mathbb{N}=\{0,1,2,\dots\}$ $\square$

2) $A=\left\{a_n=\sin\left((2n+1)\frac{\pi}{2}\right)2^{\frac{1}{n+1}} : n\geq 0\right\}=$
$=\left\{a_n=(-1)^n 2^{\frac{1}{n+1}} : n\geq 0\right\}$

Osserviamo che $b_n=2^{\frac{1}{n+1}}$ è decrescente e $\lim_{n\to+\infty}b_n=1$,
mentre $-b_n$ è crescente e $\lim_{n\to+\infty}b_n=-1$.
Poiché $a_n=b_n$ se n è pari, mentre $a_n=-b_n$ se n è dispari, risulta immediatamente che
$\underline{\sup A=\max A=a_0=2}$, mentre $\underline{\inf A=\min A=a_1=-2^{\frac{1}{2}}=-\sqrt{2}}$.



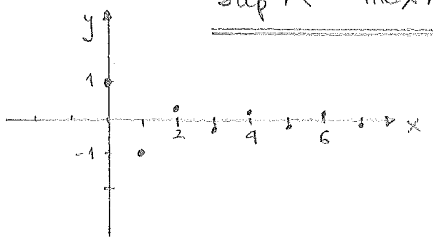
Oss. che $A=$ immagine della funzione $a_n:\mathbb{N}\to\mathbb{R}$ $\mathbb{N}=\{0,1,2,\dots\}$

$\square$

3) $A = \left\{ a_n = \frac{(-1)^n}{n^2+1} \ : \ n \geq 0 \right\}$.

Come nell'esercizio precedente si ottiene subito che

$\underline{\underline{\sup A = \max A = a_0 = 1}}$, mentre $\underline{\underline{\inf A = \min A = a_1 = -\frac{1}{2}}}$.



Oss. che $A =$ immagine della funzione $a_n : \mathbb{N} \to \mathbb{R}$,

$\mathbb{N} = \{0,1,2,\dots\}$ $\blacksquare$

<u>Es. 1.3</u> Calcolare $|z|$ nei casi $\frac{3+2i}{5-i}$ e $\frac{5+i}{3-2i}$.

i) $\frac{3+2i}{5-i} \cdot \frac{5+i}{5+i} = \frac{(15-2)+13i}{25+1} = \frac{13}{26} + \frac{13}{26}i$.

Allora $|z| = \sqrt{\left(\frac{13}{26}\right)^2 + \left(\frac{13}{26}\right)^2} = \frac{13}{26}\sqrt{2}$ $\square$

ii) $\frac{5+i}{3-2i} \cdot \frac{3+2i}{3+2i} = \frac{(15-2)+13i}{9+4} = 1+i$.

Allora $|z| = \sqrt{1+1} = \sqrt{2}$. $\blacksquare$

<u>Es. 1.4</u> Risolvere le seg. equazioni a coefficienti complessi:

1) $z + 2i = 2i\bar{z} - 1$

Poniamo $z = x+iy$. Allora $x+iy+2i = 2i(x-iy) - 1$

$\iff x+iy+2i = 2xi + 2y - 1$

$\iff (x-2y+1) + i(y-2x+2) = 0 \quad (=0+i0)$

$\iff \begin{cases} x-2y+1=0 \\ -2x+y+2=0 \end{cases} \iff \begin{cases} 2x-4y+2=0 \\ -2x+y+2=0 \end{cases}$

↑

$\iff \begin{cases} 2x-4y+2=0 \\ -3y+4=0 \end{cases}$ ← avendo sommato membro a membro le due eq. precedenti

$\Rightarrow \begin{cases} y=\frac{4}{3} \\ x=2y-1 \end{cases} \Rightarrow \begin{cases} y=\frac{4}{3} \\ x=\frac{8}{3}-1=\frac{5}{3} \end{cases}$

Infine $\underline{\underline{z = \frac{5}{3} + \frac{4}{3}i}}$. $\square$

2) $z^2 = 2i\bar{z} \iff z^2 - 2i\bar{z} = 0$.

Poniamo $z = x + iy$. Allora l'eq. si scrive

$$(x+iy)^2 - 2i(x-iy) = 0, \quad \text{ossia}$$

$$x^2 + 2xyi - y^2 - 2xi - 2y = 0.$$

Ne segue $(x^2 - 2y - y^2) + (2xy - 2x)i = 0 \qquad (= 0 + i0)$

e quindi $\begin{cases} x^2 - 2y - y^2 = 0 \\ 2x(y-1) = 0 \end{cases} \iff x = 0 \text{ o } y = 1$

Dobbiamo quindi risolvere $\begin{cases} x = 0 \\ y^2 + 2y = 0 \end{cases}$ o $\begin{cases} y = 1 \\ x^2 - 2 - 1 = 0 \end{cases}$

Abbiamo allora $\underline{\underline{z_1 = 0}}$, $\underline{\underline{z_2 = -2i}}$, $\underline{\underline{z_3 = -\sqrt{3} + i}}$, $\underline{\underline{z_4 = \sqrt{3} + i}}$. □

3) $z(\bar{z} + \operatorname{Im} z) = 3 + i$

Poniamo $z = x + iy$. Allora l'eq. si scrive

$$(x+iy)(x - iy + y) = 3 + i, \quad \text{ossia}$$

$$x^2 - \cancel{xyi} + xy + \cancel{xyi} + y^2 + y^2 i = 3 + i,$$

che possiamo scrivere anche

$$(x^2 + y^2 + xy - 3) + (y^2 - 1)i = 0 \qquad (= 0 + i0)$$

e quindi $\begin{cases} x^2 + y^2 + xy - 3 = 0 \\ y^2 - 1 = 0. \end{cases}$

Otteniamo allora dalla 2^a eq. $y_{1/2} = \pm 1$ e quindi, usando la 1^a eq.

$$\begin{cases} x^2 + 1 - x - 3 = 0 \\ y = -1 \end{cases} \quad \text{o} \quad \begin{cases} x^2 + 1 + x - 3 = 0 \\ y = 1; \end{cases}$$

si ha

$$\begin{cases} x^2 - x - 2 = 0 \\ y = -1 \end{cases} \quad \text{o} \quad \begin{cases} x^2 + x - 2 = 0 \\ y = 1. \end{cases}$$

ossia

$$\begin{cases} x_1 = -1, \; x_2 = 2 \\ y = -1 \end{cases} \quad \text{o} \quad \begin{cases} x_1 = -2, \; x_2 = 1 \\ y = 1. \end{cases}$$

Le soluzioni sono dunque $\underline{\underline{z_1 = -1 - i}}$, $\underline{\underline{z_2 = 2 - i}}$, $\underline{\underline{z_3 = -2 + i}}$ e $\underline{\underline{z_4 = 1 + i}}$. □

4) $z^2-(i+1)z+i=0$

Usiamo la formula risolutiva

$$z_{1/2}=\frac{(1+i)\pm\sqrt{(1+i)^2-4i}}{2}=\frac{(1+i)\pm\sqrt{\cancel{1}+2i-\cancel{1}-4i}}{2}$$

$$=\frac{1+i\pm\sqrt{-2i}}{2}.$$

(in realtà basterebbe calcolarne una sola!)

Dobbiamo calcolare le radici quadrate del nr. complesso:

$w=-2i=2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$. Abbiamo $w_0=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

$w_1=\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$,

ossia $w_0=\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=-1+i$

$w_1=\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=1-i$.

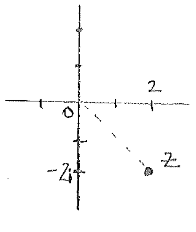
Ritornando a $z_{1/2}$ si ha $z_{1/2}=\frac{1+i\pm(-1+i)}{2}=\begin{cases}\frac{\cancel{1}+i-\cancel{1}+i}{2}\\ \frac{1+\cancel{i}+1-\cancel{i}}{2}\end{cases}$

$=\begin{cases} i \\ 1.\end{cases}$

Le soluzioni sono dunque $\underline{\underline{z_1=1}}$, $\underline{\underline{z_2=i}}$. ■

Es. 1.5 Calcolare le radici cubiche di $z=2-2i$ e $z=3+\sqrt{3}i$.

i) $z=2-2i=2\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)=2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$.



le sue radici cubiche sono allora

$$\underline{\underline{z_0=\sqrt[3]{2\sqrt{2}}\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)}}$$

$$z_1=\sqrt[3]{2\sqrt{2}}\left(\cos\left(-\frac{\pi}{12}+\frac{2\pi}{3}\right)+i\sin\left(-\frac{\pi}{12}+\frac{2\pi}{3}\right)\right)=$$

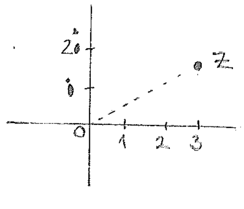
$$\underline{\underline{=\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)}}$$

$$z_2=\sqrt[3]{2\sqrt{2}}\left(\cos\left(-\frac{\pi}{12}+\frac{4\pi}{3}\right)+i\sin\left(-\frac{\pi}{12}+\frac{4\pi}{3}\right)\right)$$

$$\underline{\underline{=\sqrt[3]{2\sqrt{2}}\left(\cos\left(\frac{15\pi}{12}\right)+i\sin\left(\frac{15\pi}{12}\right)\right)}}$$

☐

ii) $z=3+\sqrt{3}i=2\sqrt{3}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$



le sue radici cubiche sono allora

$$\underline{z_0 = \sqrt[3]{2\sqrt{3}}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)}$$

$$z_1 = \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{18}+\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{18}+\frac{2\pi}{3}\right)\right)$$

$$\underline{= \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{13\pi}{18}\right) + i\sin\left(\frac{13\pi}{18}\right)\right)}$$

$$z_2 = \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{\pi}{18}+\frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{18}+\frac{4\pi}{3}\right)\right)$$

$$\underline{= \sqrt[3]{2\sqrt{3}}\left(\cos\left(\frac{25\pi}{18}\right) + i\sin\left(\frac{25\pi}{18}\right)\right)} \qquad \blacksquare$$

<u>Es. 1.6</u> Calcolare i seguenti limiti:

1) $\lim_{x\to+\infty}\left(\frac{x+1}{x+2}\right)^{3x+4} \underset{[1^\infty]}{=} \underline{\frac{1}{e^3}}$: Infatti $\left(\frac{x+1}{x+2}\right)^{3x+4} = e^{(3x+4)\log\left(1-\frac{1}{x+2}\right)}$

$$\underset{\substack{\uparrow\\ \log(1+t)=t+o(t)\text{ per }t\to0}}{=} e^{(3x+4)\left(-\frac{1}{x+2}+o\left(\frac{1}{x+2}\right)\right)} \xrightarrow[x\to+\infty]{} e^{-3} \qquad \square$$

2) $\lim_{x\to0^+}\frac{x^3+\log(1+5x^4)}{x\sin 3x} \underset{\left[\frac{0}{0}\right]}{=} 0$: Infatti $\frac{x^3+\log(1+5x^4)}{x\sin 3x} = \frac{x^3+5x^4+o(x^4)}{x(3x+o(x))}$

$$= \frac{x^3+o(x^3)}{3x^2+o(x^2)} = \frac{x+o(x)}{3+o(1)} \xrightarrow[x\to0^+]{} 0 \qquad \square$$

3) $\lim_{x\to0}\frac{e^{2x^2}-1}{e^x-1+x} \underset{\left[\frac{0}{0}\right]}{=} \underline{0}$: Infatti $\frac{e^{2x^2}-1}{e^x-1+x} = \frac{2x^2+o(x^2)}{2x+o(x)} \longrightarrow 0$ per $x\to0$ $\square$

4) $\lim_{x\to0}\frac{e^x-1+x^3}{1-\cos 2x} \underset{\left[\frac{0}{0}\right]}{=} \nexists$: Infatti $\frac{e^x-1+x^3}{1-\cos 2x} = \frac{x+o(x)}{\frac{(2x)^2}{2}+o(x^2)} = \frac{1+o(1)}{2x+o(x)} \rightarrow \nexists$ $\square$

5) $\lim_{x\to\frac{1}{2}^+}\frac{\log\left(1+\sqrt{2x+1}\right)}{\sqrt{4x^2-1}} = \underline{+\infty}$ $\square$

$$\log\left(1+\sqrt{2x+1}\right) \rightarrow \log(1+\sqrt{2})$$

$$\sqrt{4x^2-1} \rightarrow \text{infinitesimo positivo}$$

6) $\lim_{x\to0^+}\frac{\sin x^2-\log(1+3x)}{\sin x^4+\log(1+x^4)} \underset{\left[\frac{0}{0}\right]}{=} \underline{-\infty}$: Infatti $\frac{\sin x^2-\log(1+3x)}{\sin x^4+\log(1+x^4)} =$

$$\frac{x^2+o(x^2)-3x+o(x)}{x^4+o(x^4)+x^4+o(x^4)} = \frac{-3x+o(x)}{2x^4+o(x^4)}$$

$$= \frac{-3+o(1)}{2x^3+o(x^3)} \qquad \square$$

7) $\lim_{n\to+\infty} \sqrt[n]{(n^2+1)\sin\left(\frac{1}{n}\right)} = 1$ $\left[\infty^0\right]$

Infatti $(n^2+1)\sin\left(\frac{1}{n}\right) = (n^2+1)\left(\frac{1}{n} + o\left(\frac{1}{n^2}\right)\right)$ $n\to+\infty$

$= n + \frac{1}{n} + o(1) + o\left(\frac{1}{n^2}\right)$ per $n\to+\infty$

$= n + o(1)$ per $n\to+\infty$. Quindi

$$\sqrt[n]{(n^2+1)\sin\left(\frac{1}{n}\right)} = e^{\frac{1}{n}\log(n+o(1))} \xrightarrow[n\to+\infty]{} e^0 = 1. \quad \square$$

8) $\lim_{n\to+\infty} \frac{2+(-1)^n}{3^n} = 0$: Infatti $\underset{\searrow 0}{\frac{1}{3^n}} < \frac{2+(-1)^n}{3^n} \leq \underset{\swarrow 0}{\frac{3}{3^n}}$ $n\to+\infty$, e

per il teorema del confronto (dei 2 carabinieri)

si ha che $\lim_{n\to+\infty} \frac{2+(-1)^n}{3^n} = 0$. $\square$

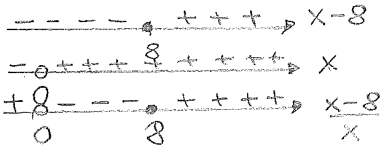
9) $\lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{-n} = e$: $e^{-n\log\left(1-\frac{1}{n}\right)} = e^{-n\left(-\frac{1}{n}+o\left(\frac{1}{n}\right)\right)} = e^{1+o(1)} \xrightarrow[n\to+\infty]{} e.$

10) $\lim_{n\to+\infty} \left(1+\frac{1}{n^3}\right)^{n^2} = 1.$: $e^{n^2\log\left(1+\frac{1}{n^3}\right)} = e^{n^2\left(\frac{1}{n^3}+o\left(\frac{1}{n^3}\right)\right)} \xrightarrow[n\to+\infty]{} e^0$ ▨

Es. 1.7 Studiare le seguenti funzioni e tracciarne il grafico.

1) $\boxed{f(x) = \sqrt{x + \frac{8}{x} + \left|1 - \frac{8}{x}\right|}}$

- Notiamo che $1 - \frac{8}{x} = \frac{x-8}{x}$

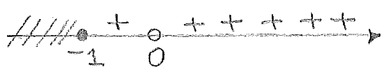


quindi
$$f(x) = \begin{cases} \sqrt{x + \cancel{\frac{8}{x}} + 1 - \cancel{\frac{8}{x}}} & \text{se } -1 \leq x < 0 \text{ o } x \geq 8 \\ \sqrt{x + \frac{8}{x} - 1 + \frac{8}{x}} & \text{se } 0 < x < 8 \end{cases}$$

Nota: $x - 1 + \frac{16}{x} = \frac{x^2 - x + 16}{x} > 0$ $\forall x > 0$

Abbiamo $\text{dom} f = \underline{\underline{[-1, +\infty[\setminus \{0\}}}$

- segno di f



- $\lim_{x\to0^-} f(x) = 1$, $\lim_{x\to0^+} f(x) = +\infty$ $x=0$ asintoto verticale (destro) per f

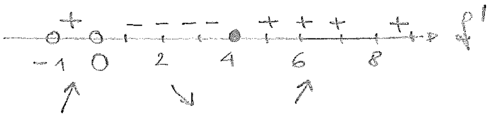
$\lim_{x\to+\infty} f(x) = +\infty$.

- f continua su dom f
- dom f' = dom f \ {-1, 8}

$$f'(x) = \begin{cases} \frac{1}{2\sqrt{x+1}} & \text{se } -1<x<0 \quad \circ \quad x>8 \\ \frac{1}{2\sqrt{\frac{x^2-x+16}{x}}} \cdot \frac{x(2x-1)-(x^2-x+16)}{x^2} & \text{se } 0<x<8 \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{x+1}} & \text{se } -1<x<0 \quad \circ \quad x>8 \\ \frac{1}{2\sqrt{\frac{x^2-x+16}{x}}} \cdot \frac{x^2-16}{x^2} & \text{se } 0<x<8 \end{cases}$$

$f'(x) = 0 \iff x = 4$



x = 4 pt. di min. loc. stretto per f

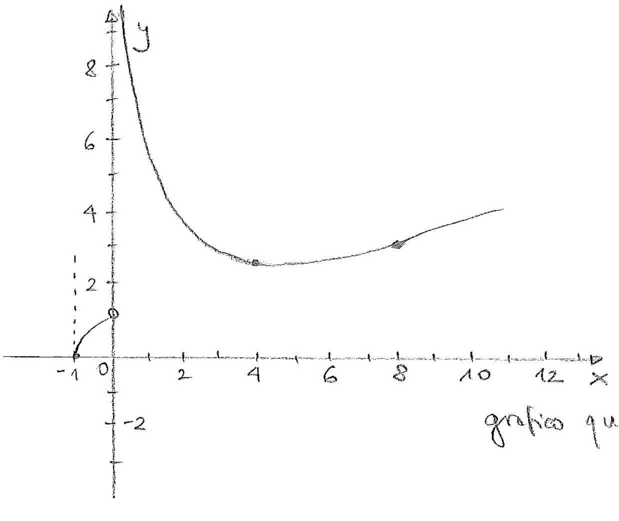
$$f(4) = \sqrt{\frac{28}{4}} = \sqrt{7} \sim 2,6$$

Abbiamo $f'_+(-1) = \lim_{x \to -1^+} f'(x) = +\infty$ $\qquad \lim_{x \to 0^-} f'(x) = \frac{1}{2}$

$$f'_-(8) = \frac{1}{2\sqrt{\frac{64-8+16}{8}}} \cdot \frac{64-16}{64} = \frac{\overset{24}{\cancel{48}}}{64} \cdot \frac{1}{\cancel{2}} \cdot \sqrt{\frac{8}{\cancel{72}_9}} = $$

$$= \frac{\cancel{24}^{8}}{\cancel{3} \cdot \cancel{64}_8} = \frac{1}{8}$$

$$f'_+(8) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$



x=8 punto angoloso

grafico qualitativo di f.

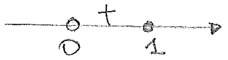
□

2) $$\boxed{f(x) = \begin{cases} x\sqrt{\frac{1}{|x|}-1} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}}$$

se $x \neq 0$ $\quad f(x) = x\sqrt{\frac{1-|x|}{|x|}}$ e quindi è ben definita solo se $|x| \leq 1$.

Oss. inoltre che f è dispari su $[-1,1]$ e quindi basta studiarla per $x \in]0,1]$.

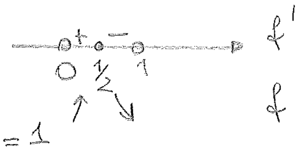
Abbiamo $f(x) \geq 0$ su $]0,1]$



Inoltre $f(x) = \sqrt{\frac{x^{\cancel{2}}(1-x)}{\cancel{x}}} = \sqrt{x-x^2}$

- Si ha allora $\lim_{x\to 0^+} f(x) = 0 = f(0)$. Risulta che f è continua su $[0,1]$.

- dom $f' =]0,1[$ $\qquad f'(x) = \frac{1-2x}{2\sqrt{x-x^2}}$



$x = \frac{1}{2}$ pt. critico per f e $f\left(\frac{1}{2}\right) = \sqrt{\frac{1}{2}-\frac{1}{4}} = \frac{1}{2}$.

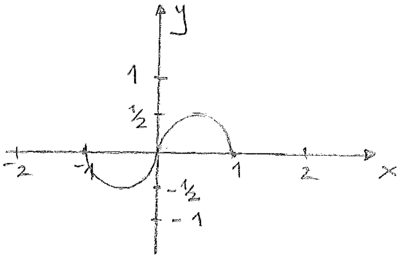
Inoltre $f'_-(1) = \lim_{x\to 1^-} f'(x) = -\infty$. $\qquad f'_+(0) = \lim_{x\to 0^+} f'(x) = +\infty$

- dom $f'' =]0,1[$

$$f''(x) = \frac{1}{2}\,\frac{-2\sqrt{x-x^2} - (1-2x)\frac{1-2x}{2\sqrt{x-x^2}}}{(x-x^2)} =$$

$$= \frac{1}{2}\,\frac{-4(x-x^2)-(1-2x)^2}{2(x-x^2)^{3/2}} = \frac{1}{4}\,\frac{\cancel{-4x}+\cancel{4x^2}-1+\cancel{4x}-\cancel{4x^2}}{(x-x^2)^{3/2}}$$

$$= \frac{-1}{(x-x^2)^{3/2}}$$



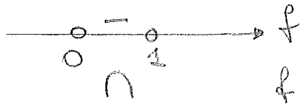


grafico qualitativo di f

$\square$

(R1)

3) $\boxed{f(x) = e^x - 2e^{10}|x-10|}$

- dom $f = \mathbb{R}$
- $f(x) = \begin{cases} e^x - 2e^{10}x + 20e^{10} & \text{se } x \geq 10 \\ e^x + 2e^{10}x - 20e^{10} & \text{se } x < 10 \end{cases}$
- $\lim_{x \to -\infty} f(x) = -\infty$ $\lim_{x \to +\infty} f(x) = +\infty$; f è continua su $\mathbb{R}$.
- dom $f' = \mathbb{R} \setminus \{10\}$ $f'(x) = \begin{cases} e^x - 2e^{10} & \text{se } x > 10 \\ e^x + 2e^{10} & \text{se } x < 10 \end{cases}$.

$y = 2e^{10}x - 20e^{10}$ asintoto obliq. per f, per $x \to -\infty$

$$f'_-(10) = \lim_{x \to 10^-} f'(x) = 3e^{10}$$
$$f'_+(10) = \lim_{x \to 10^+} f'(x) = -e^{10}$$

$x = 10$ pt. angoloso per f.

$e^x - 2e^{10} = 0 \Leftrightarrow e^x = 2e^{10}$
$x = \log(2e^{10})$
$= 10 + \log 2$

$f(10 + \log 2) = 2e^{10} - 2e^{10}\log 2$
$= 2e^{10}(1 - \log 2) \sim 0.6e^{10}$

$+ + + + + \circ - \bullet + + + \rightarrow$ $f'(x)$
10 $10 + \log 2$
$\nearrow$ $\searrow$ $\nearrow$ $f(x)$

- dom $f'' = \mathbb{R} \setminus \{10\}$ $f''(x) = e^x$ $\forall x \neq 10$

$+ + + + \circ + + + + \rightarrow$ $f''(x)$
10
$\cup$ $\cup$ $f(x)$

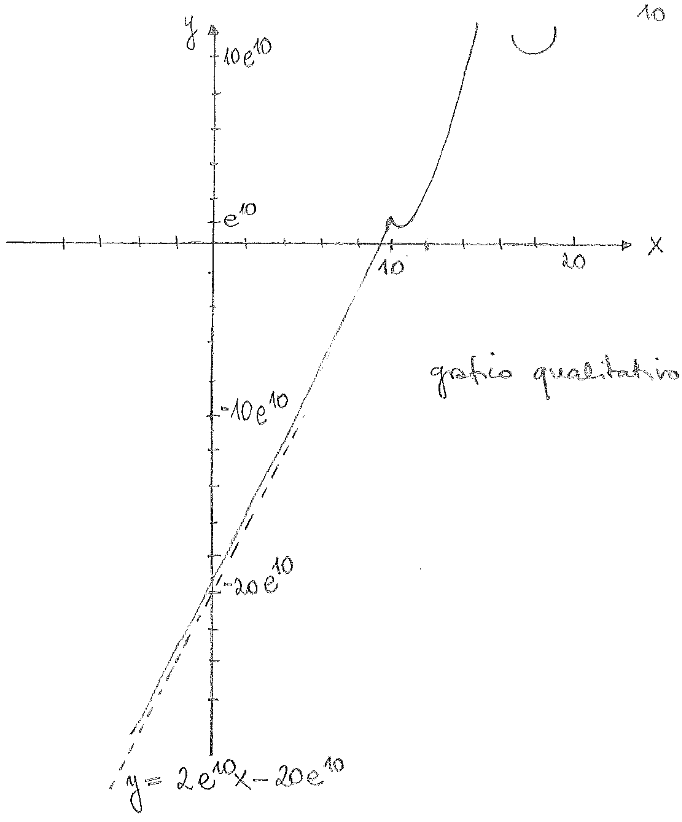


grafico qualitativo di f

□

R1

4) $\boxed{f(x) = 5x^2 - 90\,|\log x|}$

- $\operatorname{dom} f =]0, +\infty[$
- $\lim_{x \to 0^+} f(x) = -\infty \qquad \lim_{x \to +\infty} f(x) = +\infty$

$\left(\text{poiché } \frac{|\log x|}{x^2} \xrightarrow[x \to +\infty]{} 0\right)$

- $f(x) = \begin{cases} 5x^2 - 90 \log x & \text{se } x \geq 1 \\ 5x^2 + 90 \log x & \text{se } 0 < x < 1 \end{cases}$

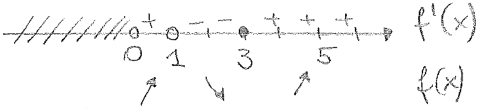
$\to$ $x=0$ asintoto verticale (destro) per f

- f è continua in dom f
- $\operatorname{dom} f' =]0, +\infty[\setminus \{1\}$

$$f'(x) = \begin{cases} 10x - \frac{90}{x} & \text{se } x > 1 \\ 10x + \frac{90}{x} & \text{se } 0 < x < 1 \end{cases} = \begin{cases} \frac{10(x^2-9)}{x} & \text{se } x > 1 \\ \frac{10(x^2+9)}{x} & \text{se } 0 < x < 1 \end{cases}$$

$x = 3$ pto critico per f

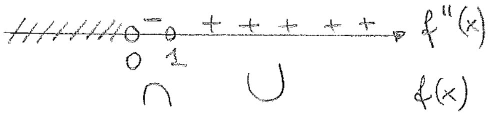
$$f(3) = 45 - 90 \log 3 = 45(1 - 2\log 3) \sim -54$$



$f'_-(1) = 100 \qquad f'_+(1) = -80$

- $\operatorname{dom} f'' =]0, +\infty[\setminus \{1\}$

$$f''(x) = \begin{cases} 10 + \frac{90}{x^2} & \text{se } x > 1 \\ 10 - \frac{90}{x^2} & \text{se } 0 < x < 1 \end{cases} = \begin{cases} \frac{10(x^2+9)}{x^2} & \text{se } x > 1 \\ \frac{10(x^2-9)}{x^2} & \text{se } 0 < x < 1 \end{cases}$$



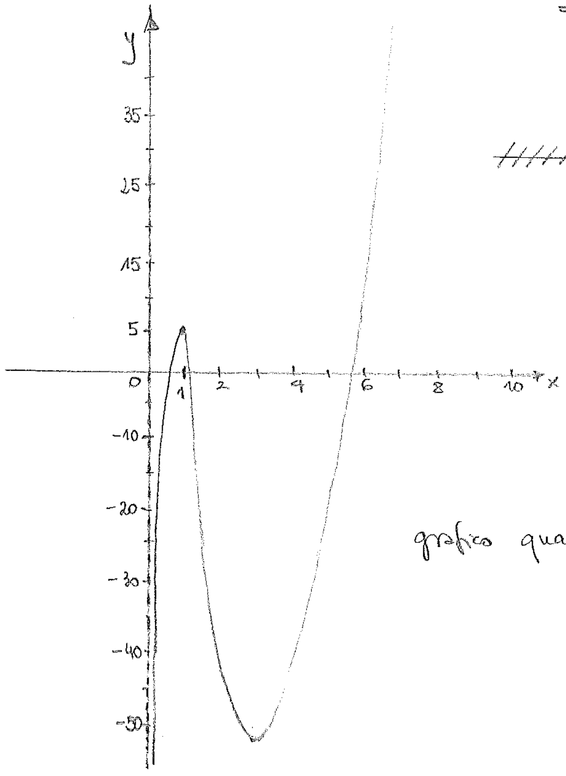


grafico qualitativo di f

☐

5) $\boxed{f(x) = x + \operatorname{arctg}\left(\frac{2x+1}{x}\right)}$

- $\operatorname{dom} f = \mathbb{R}\setminus\{0\} =]-\infty, 0[\, \cup \,]0, +\infty[$
- $\lim_{x\to-\infty} f(x) = -\infty \qquad \lim_{x\to 0^-} f(x) = -\frac{\pi}{2}$

  $\lim_{x\to 0^+} f(x) = \frac{\pi}{2} \qquad \lim_{x\to+\infty} f(x) = +\infty$

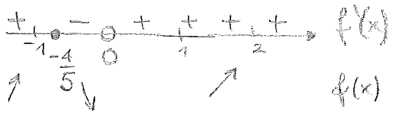
Notiamo che $\lim_{\substack{x\to-\infty \\ (+\infty)}} \frac{f(x)}{x} = 1$, $\lim_{\substack{x\to-\infty \\ (+\infty)}} \left[f(x) - x\right] = \operatorname{arctg} 2$. Risulta che

$y = x + \operatorname{arctg} 2$ è asintoto orizzontale per f, per $x \to \underset{(+\infty)}{-\infty}$.

- f è continua nel suo dominio.
- $\operatorname{dom} f' = \mathbb{R}\setminus\{0\}$ $\qquad f'(x) = 1 + \frac{1}{1+\left(\frac{2x+1}{x}\right)^2} \cdot \frac{2\cdot x - (2x+1)}{x^2} = 1 + \frac{-1}{5x^2+4x+1}$

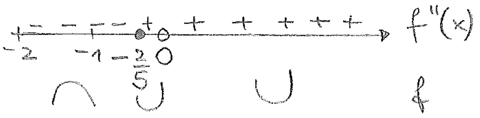
$$= \frac{5x^2+4x}{5x^2+4x+1} \qquad \lim_{x\to 0^+} f'(x) = 0 = \lim_{x\to 0^-} f'(x)$$

$x = -\frac{4}{5}$ è pt. di max. loc. stretto per f



$f\left(-\frac{4}{5}\right) = -\frac{4}{5} + \operatorname{arctg}\left(\frac{-3/5}{-4/5}\right) = -\frac{4}{5} + \operatorname{arctg}\left(\frac{3}{4}\right) < 0 \qquad \left(-\frac{4}{5} + \operatorname{arctg}\left(\frac{3}{4}\right) < -\frac{4}{5} + \operatorname{arctg} 1 = -\frac{4}{5} + \frac{\pi}{4} < 0\right)$

- $\operatorname{dom} f'' = \mathbb{R}\setminus\{0\}$ $\qquad f''(x) = \frac{10x+4}{(5x^2+4x+1)^2} = \frac{2(5x+2)}{(5x^2+4x+1)^2}$



$x = -\frac{2}{5}$ pt. di flesso per f

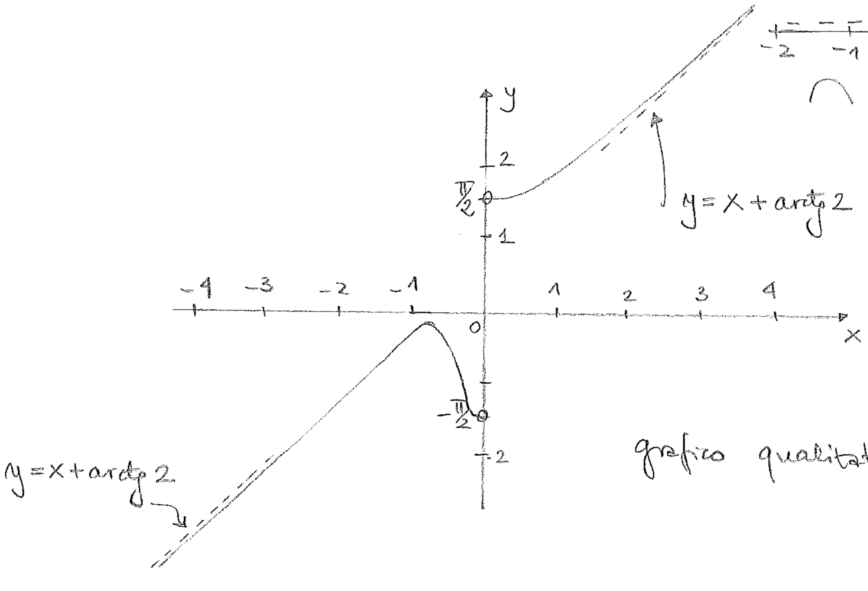


grafico qualitativo di f

□

6) $\boxed{f(x) = \operatorname{arctg} x - \frac{1}{5}|\sqrt{5} + x|}$

- $\operatorname{dom} f = \mathbb{R} =]-\infty, +\infty[$
- $\lim_{x \to -\infty} f(x) = -\infty \qquad \lim_{x \to +\infty} f(x) = -\infty$
- f è continua in $\mathbb{R}$

- $$f(x) = \begin{cases} \operatorname{arctg} x - \frac{x}{5} - \frac{\sqrt{5}}{5} & \text{se } x \geq -\sqrt{5} \\ \operatorname{arctg} x + \frac{x}{5} + \frac{\sqrt{5}}{5} & \text{se } x < -\sqrt{5} \end{cases}$$

$y = -\frac{x}{5} - \frac{\sqrt{5}}{5} + \frac{\pi}{2}$ asint. obliquo per f, per $x \to +\infty$

$y = \frac{x}{5} + \frac{\sqrt{5}}{5} - \frac{\pi}{2}$ asintoto obliquo per f, per $x \to -\infty$

$\operatorname{dom} f' = \mathbb{R} \setminus \{-\sqrt{5}\}$

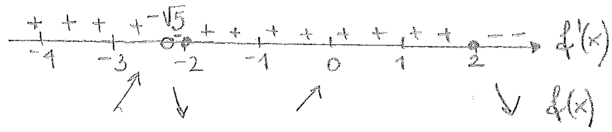
$$f'(x) \overset{⊛}{=} \begin{cases} \frac{1}{1+x^2} - \frac{1}{5} & \text{se } x > -\sqrt{5} \\ \frac{1}{1+x^2} + \frac{1}{5} & \text{se } x < -\sqrt{5} \end{cases}$$

$$f'_-(-\sqrt{5}) = \lim_{x \to -\sqrt{5}^-} f'(x) = \frac{1}{6} + \frac{1}{5} = \frac{11}{30}$$

$$f'_+(-\sqrt{5}) = \lim_{x \to -\sqrt{5}^+} f'(x) = \frac{1}{6} - \frac{1}{5} = -\frac{1}{30}$$

$x = -\sqrt{5}$ è un pt. angoloso per f

$$f'(x) = \begin{cases} \frac{4 - x^2}{5(1+x^2)} & \text{se } x > -\sqrt{5} \\ \frac{6 + x^2}{5(1+x^2)} & \text{se } x < -\sqrt{5} \end{cases}$$



- $\operatorname{dom} f'' = \mathbb{R} \setminus \{-\sqrt{5}\}$

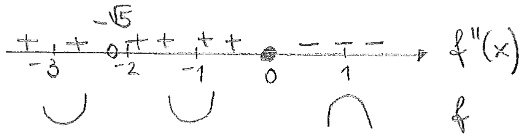
$x = \pm 2$ pt. critici per f

$x = -2$ pt. di min. loc. stretto per f

$x = 2$ pt. di max. loc. stretto per f

$$f''(x) \overset{⊛'}{=} \begin{cases} \frac{1}{5} \frac{-2x(1+x^2) - (4-x^2)2x}{(1+x^2)^2} & x > -\sqrt{5} \\ \frac{1}{5} \frac{2x(1+x^2) - (6+x^2)2x}{(1+x^2)^2} & x < -\sqrt{5} \end{cases}$$

$$= \begin{cases} \frac{1}{\cancel{5}} \frac{-\cancel{10}^{2}x}{(1+x^2)^2} & x > -\sqrt{5} \\ \frac{1}{\cancel{5}} \frac{-\cancel{10}^{2}x}{(1+x^2)^2} & x < -\sqrt{5} \end{cases}$$



$x = 0$ pt. di flesso per f

$f(0) = -\frac{\sqrt{5}}{5}$.

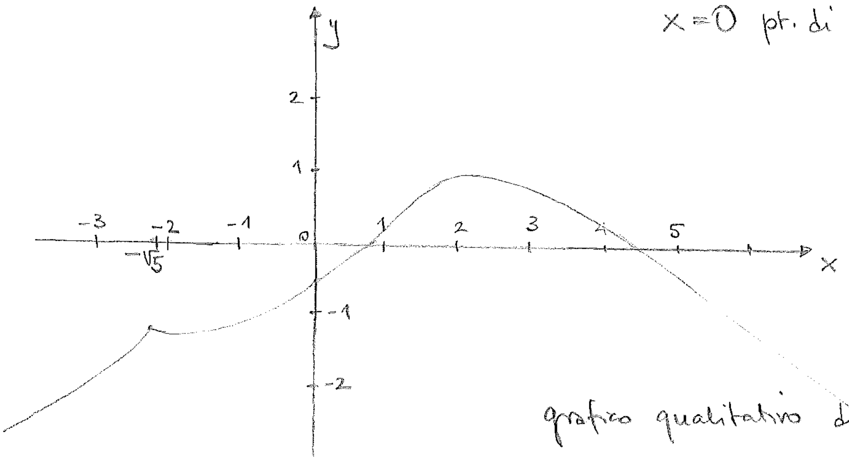


grafico qualitativo di f

□

NOTA: ⊛' L'espressione di $f''(x)$ si ottiene molto più facilmente usando l'espressione di $f'(x)$ in ⊛.

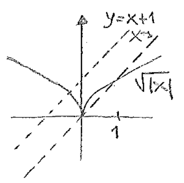
7) $\boxed{f(x)=\log\left(\sqrt{|x|}-x\right)}$

• $\operatorname{dom} f = \{x\in\mathbb{R} : \sqrt{|x|}-x>0\}$

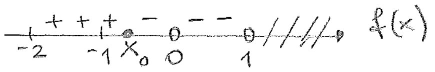
$= \{x\in\mathbb{R} : \sqrt{|x|}>x\}$

$= \left]-\infty,1\right[\setminus\{0\}$

$= \left]-\infty,0\right[\cup\left]0,1\right[$



• $f(x)=0 \iff \sqrt{|x|}-x=1$ $\exists x_0\in\left]-1,0\right[: f(x_0)=0$



• $\lim_{x\to-\infty} f(x)=+\infty$

$\lim_{x\to0^-} f(x)=-\infty$ $x=0$ asintoto verticale (sinistro) di f

$\lim_{x\to0^+} f(x)=-\infty$ $x=0$ asintoto verticale (destro) di f

$\lim_{x\to1^-} f(x)=-\infty$ $x=1$ asintoto verticale (sinistro) di f.

• f è continua nel suo dominio.

• $\operatorname{dom} f' = \operatorname{dom} f$

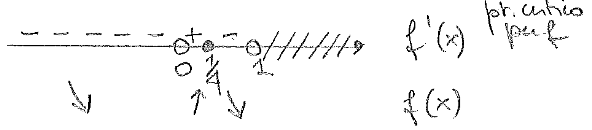
$$f(x)=\begin{cases}\log\left(\sqrt{-x}-x\right) & \text{se } x<0\\ \log\left(\sqrt{x}-x\right) & \text{se } 0<x<1\end{cases}$$

$$f'(x)=\begin{cases}\dfrac{\dfrac{-1}{2\sqrt{-x}}-1}{\sqrt{-x}-x} & \text{se } x<0\\[2ex] \dfrac{\dfrac{1}{2\sqrt{x}}-1}{\sqrt{x}-x} & \text{se } 0<x<1\end{cases}$$

$$=\begin{cases}\dfrac{-1-2\sqrt{-x}}{2\sqrt{-x}\left(\sqrt{-x}-x\right)} & \text{se } x<0\\[2ex] \dfrac{1-2\sqrt{x}}{2\sqrt{x}\left(\sqrt{x}-x\right)} & \text{se } 0<x<1\end{cases}$$

Risulta che $f'(x)=0 \iff 1-2\sqrt{x}=0$ con $0<x<1$. si ha $x=\frac{1}{4}$

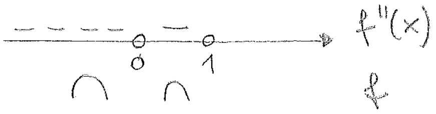
$f\left(\frac{1}{4}\right)=\log\left(\frac{1}{2}-\frac{1}{4}\right)=-\log 4$
~ -1.4



• $\operatorname{dom} f'' = \operatorname{dom} f$

$$f''(x)=\begin{cases}\frac{1}{2}\left[\dfrac{-1+2x-\frac{5}{2}\sqrt{-x}}{-x\left(\sqrt{-x}-x\right)^2}\right] & \text{se } x<0\\[2ex] \frac{1}{2}\left[\dfrac{-1-2x+\frac{5}{2}\sqrt{x}}{x\left(\sqrt{x}-x\right)^2}\right] & \text{se } 0<x<1\end{cases}$$

Risulta quindi



(Nota: $-1-2x+\frac{5}{2}\sqrt{x}$ risulta negativa poiché è una funzione crescente su $]0,1[$ e $\lim_{x\to 1^-}\left(-1-2x+\frac{5}{2}\sqrt{x}\right) = -\frac{1}{2} < 0$; si può anche osservare che $-2x+\frac{5}{2}\sqrt{x}-1 \underset{\overset{\uparrow}{t=\sqrt{x}}}{=} -2t^2+\frac{5}{2}t-1$ che non ammette radici reali ed è quindi sempre negativa.

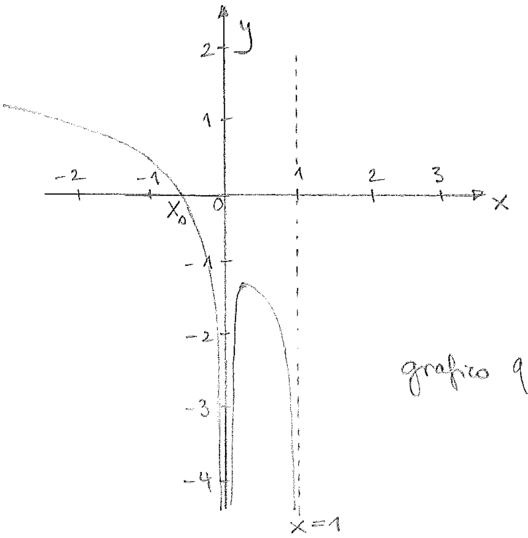


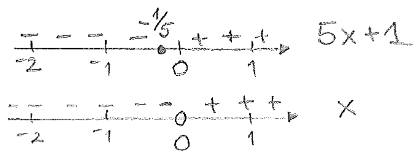
grafico qualitativo di f

□

8) $\boxed{f(x)= |x| + \log\left(\frac{5x+1}{x}\right)}$

- $\text{dom} f = \left\{x\in\mathbb{R} : \frac{5x+1}{x} > 0\right\}$

  $=]-\infty, -\frac{1}{5}[\cup]0,+\infty[$



- $\lim_{x\to -\infty} f(x) = +\infty$

  $\lim_{x\to -\frac{1}{5}^-} f(x) = -\infty$ $x = -\frac{1}{5}$ asintoto verticale (sinistro) per f

  $\lim_{x\to 0^+} f(x) = +\infty$ $x = 0$ asintoto verticale (destro) per f

  $\lim_{x\to +\infty} f(x) = +\infty$

Notiamo che $\lim_{x\to -\infty} \frac{f(x)}{x} = \lim_{x\to -\infty} \frac{-x}{x} = -1$ e
$\lim_{x\to -\infty} \left[f(x)+x\right] = \lim_{x\to -\infty} \log\left(\frac{5x+1}{x}\right) = \log 5$ $\Bigg\}$ $\Rightarrow$ $y=-x+\log 5$ asintoto obliquo per f, per $x\to-\infty$.

Analogamente risulta che $y = x+\log 5$ è asintoto obliquo per f, per $x\to+\infty$.

(R1)

- f è continua nel suo dominio.
- dom f' = dom f $\qquad f(x)=\begin{cases} -x+\log\left(\frac{5x+1}{x}\right) & \text{se } x<-1/5 \\ x+\log\left(\frac{5x+1}{x}\right) & \text{se } x>0 . \end{cases}$

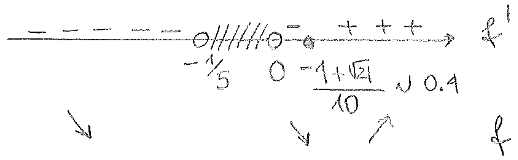
Risulta

$$f'(x)=\begin{cases} -1+\frac{\cancel{x}}{5x+1}\cdot\frac{5x-(5x+1)}{x^{\cancel{2}}} & \text{se } x<-1/5 \\ 1+\frac{\cancel{x}}{5x+1}\cdot\frac{5x-(5x+1)}{x^{\cancel{2}}} & \text{se } x>0 \end{cases}$$

$$f'(x)=\begin{cases} -1-\frac{1}{5x^2+x} & \text{se } x<-1/5 \\ 1-\frac{1}{5x^2+x} & \text{se } x>0 \end{cases}$$

$$=\begin{cases} \frac{-5x^2-x-1}{5x^2+x} & \text{se } x<-1/5 \\ \frac{5x^2+x-1}{5x^2+x} & \text{se } x>0 \end{cases}$$

$x=\frac{-1+\sqrt{21}}{10}$ pt. di min. loc. stretto per f



- dom f'' = dom f $\qquad f''(x)=\begin{cases} \frac{10x+1}{(5x^2+x)^2} & \text{se } x<-1/5 \\ \frac{10x+1}{(5x^2+x)^2} & \text{se } x>0 \end{cases}$

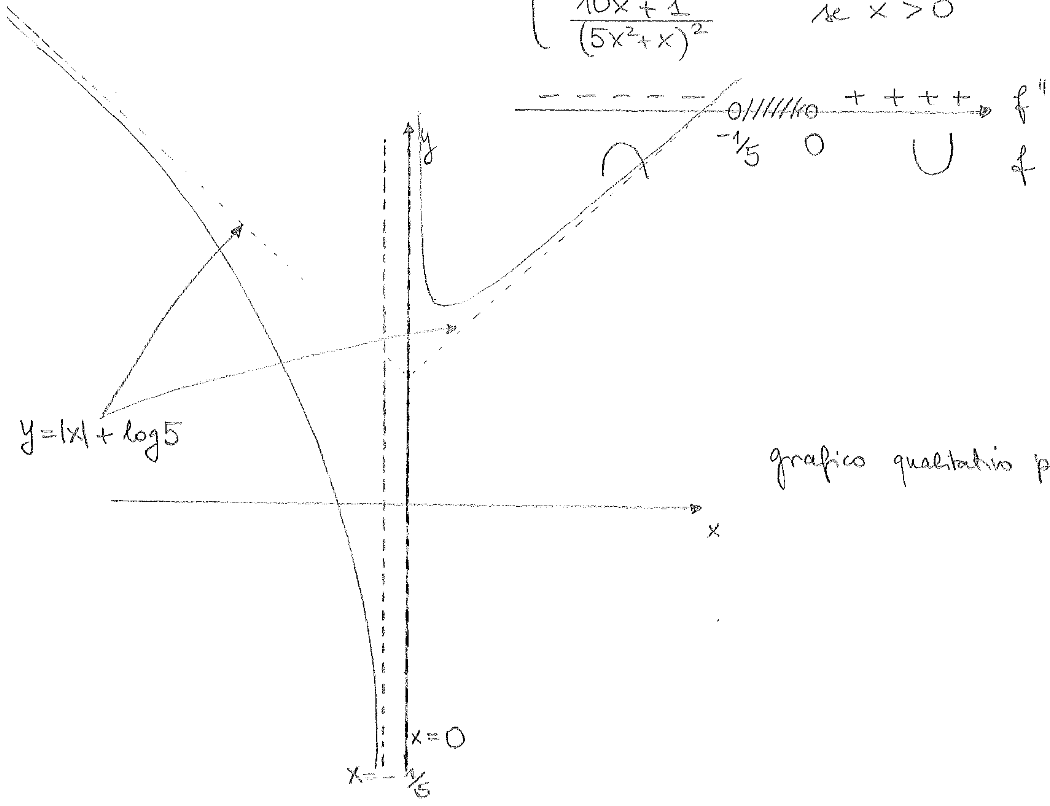


grafico qualitativo per f