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1) a) $f(x)=\begin{cases}\int_0^x (t^2+2)\,dt & \text{se } x>0\\ a^2x+b & \text{se } x\leq 0.\end{cases}$ (nota: volendo si può integrare!)

Abbiamo che f è continua e derivabile $\forall x\neq 0$. Dobbiamo solo richiedere la continuità e derivabilità in $x=0$.

f è continua in $x=0 \iff \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$.

Per questo deve essere $\underline{0=b}$.

Notiamo che per il TFC(*) si ha $f'(x)=x^2+2$ se $x>0$. D'altra parte $f'(x)=a^2$ se $x<0$. Poiché f è continua e $f'_+(0)=\lim_{x\to 0} f'(x)=2$, $f'_-(0)=\lim_{x\to 0^-} f'(x)=a^2$, f è derivabile in $x=0$ se e solo se $a^2=2$, quindi $\underline{\underline{a=\pm\sqrt{2}}}$. □

b) $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\int_0^{x^2}\cos 2t\,dt$. Il TFC garantisce che g è derivabile e $g'(x)=(\cos 2x^2)\cdot 2x$. Quindi $g'(x)=0 \iff x=0,\ 2x^2=\frac{\pi}{2}+k\pi \quad k\geq 0$.

Quindi $x^2=\frac{\pi}{4}+k\frac{\pi}{2} \quad k\geq 0$. L'insieme dei pt. critici è quindi costituito da $x=0$ e $x=\pm\sqrt{\frac{\pi}{4}+\frac{k\pi}{2}}$, $k\geq 0$.

Abbiamo $\lim_{x\to 0^+}\frac{g(x)-x^2}{x^6}$ $\left[\frac{0}{0}\right]$. Possiamo applicare il teorema di de l'Hopital. Osserviamo che $\lim_{x\to 0^+}\frac{g'(x)-2x}{6x^5}=\lim_{x\to 0^+}\frac{2x(\cos 2x^2)-2x}{6x^5}$

$$=\lim_{x\to 0^+}\frac{\cos 2x^2-1}{3x^4}=\lim_{x\to 0^+}\underbrace{\frac{\cos 2x^2-1}{4x^4}}_{\text{limite notevole } -\frac{1}{2}}\cdot\frac{4}{3}=-\frac{2}{3}.$$

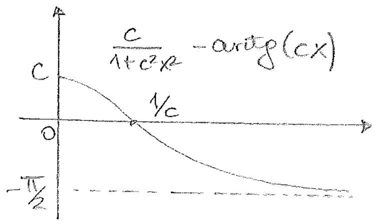
Possiamo allora concludere che $\lim_{x\to 0^+}\frac{g(x)-x^2}{x^6}=\underline{\underline{-\frac{2}{3}}}$ ■

2) i) Per il TFC si ha $F'_c(x)=e^{-x}\operatorname{arctg}(cx)$, quindi

$$F''_c(x)=e^{-x}\left[\frac{c}{1+c^2x^2}-\operatorname{arctg}(cx)\right].$$

(*) TFC = Teorema Fondamentale del Calcolo Integrale

Poiché $c > 0$, tale derivata seconda è certamente positiva per $x \le 0$, perché $\operatorname{arctg}(cx) \le 0$. Poi, per $x > 0$, la funzione $\frac{c}{1+c^2x^2} - \operatorname{arctg}(cx)$ è decrescente (somma di due funzioni decrescenti): per vedere se è non-negativa per $0 < x \le 1/c$ basta vedere se lo è nel pt. $1/c$; in tal punto, la derivata seconda vale



$$e^{-1/c}\left[\frac{c}{2} - \operatorname{arctg} 1\right] = \frac{e^{-1/c}}{2}\left(c - \frac{\pi}{2}\right).$$

Quindi, la funzione F_c è convessa su $\left]-\infty, \frac{1}{c}\right]$ se e solo se $\boxed{c \geq \frac{\pi}{2}}$

□

ii) Poiché $\lim_{x\to 0^+} e^{-x} = 1$, il limite da calcolare è $\lim_{x\to 0^+} \frac{F_c(x)}{x^2}$.

Poiché il limite si presenta nella forma $\frac{0}{0}$ e tutte le ipotesi del teorema di de l'Hôpital sono soddisfatte, calcoliamo

$\lim_{x\to 0^+} \frac{F_c'(x)}{2x} = \lim_{x\to 0^+} \frac{e^{-x}\operatorname{arctg}(cx)}{2x} = \frac{c}{2}$. Possiamo concludere che il limite dato risulta $\lim_{x\to 0^+} \frac{F_c(x)}{x^2} = \boxed{\frac{c}{2}}$.

■

3) i) $\int \sqrt[3]{2x+1}\,dx = \frac{1}{2}\int 2\,(2x+1)^{1/3}\,dx = \frac{1}{2}\,\frac{(2x+1)^{4/3}}{4/3} + c = \underline{\underline{\frac{3}{8}\sqrt[3]{(2x+1)^4} + c}}.$

ii) $\int x^2(3+x^3)^{-1/5}\,dx = \frac{1}{3}\int 3x^2(3+x^3)^{-1/5}\,dx = \frac{1}{3}\,\frac{(3+x^3)^{4/5}}{4/5} + c = \underline{\underline{\frac{5}{12}\sqrt[5]{(3+x^3)^4} + c}}.$

iii) $\int \frac{3-4x}{1+x^2}\,dx = 3\int \frac{1}{1+x^2}\,dx - \frac{4}{2}\int \frac{2x}{1+x^2}\,dx = \underline{\underline{3\operatorname{arctg} x - 2\log(1+x^2) + c}}$

iv) $\int \frac{1-\sin x}{x+\cos x}\,dx = \underline{\underline{\log|x+\cos x| + c}}.$

v) $\int \frac{-1}{x\sqrt{1-\log^2 x}}\,dx = \underline{\underline{\arccos(\log x) + c}}.$

vi) $\int 4xe^{-x^2}\,dx = -2\int -2xe^{-x^2}\,dx = \underline{\underline{-2\,e^{-x^2} + c}}.$

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vii) $\int (\cos x)^{1/3} \sin x\, dx = -\int (\cos x)^{1/3}(-\sin x)\, dx = -\frac{(\cos x)^{4/3}}{4/3} + c = \underline{-\frac{3}{4}(\cos x)^{4/3} + c}.$

viii) $\int \frac{4e^x}{1+e^{2x}}\, dx = 4\int \frac{e^x}{1+e^{2x}}\, dx = \underline{4 \operatorname{arctg}(e^x) + c}.$

ix) $\int \frac{\sin 2x}{\cos 2x}\, dx = -\frac{1}{2}\int \frac{-2\sin 2x}{\cos 2x}\, dx = \underline{-\frac{1}{2}\log|\cos 2x| + c}.$

x) $\int \frac{e^{2x}-1}{e^x+1}\, dx = \int \frac{(e^x-1)\cancel{(e^x+1)}}{\cancel{e^x+1}}\, dx = \underline{e^x - x + c}.$

xi) $\int \frac{1}{3x^2+2}\, dx = \frac{1}{2}\int \frac{1}{1+\frac{3}{2}x^2}\, dx = \frac{1}{\sqrt{3}\sqrt{2}}\int \frac{\sqrt{\frac{3}{2}}}{1+\left(\sqrt{\frac{3}{2}}x\right)^2}\, dx = \underline{\frac{1}{\sqrt{6}} \operatorname{arctg}\left(\sqrt{\frac{3}{2}}x\right) + c}.$

xii) $\int \frac{2^x}{\cos^2(2^x+3)}\, dx = \frac{1}{\log 2}\int \frac{2^x \log 2}{\cos^2(2^x+3)}\, dx = \underline{\frac{1}{\log 2}\operatorname{tg}(2^x+3) + c}.$

4) i) $\int \arccos x = \int 1\cdot \arccos x\, dx = x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\, dx$

$= x\arccos x + \int x(1-x^2)^{-1/2}\, dx = \underline{x\arccos x - (1-x^2)^{1/2} + c}.$

ii) $\int x^3 \log^2 x\, dx = \frac{x^4}{4}\log^2 x - \int \frac{\cancel{x^4}^{3}}{\cancel{4}_{2}}\cdot \cancel{2}\log x \cdot \frac{1}{\cancel{x}}\, dx =$

$= \frac{x^4}{4}\log^2 x - \frac{1}{2}\int x^3 \log x\, dx = \frac{x^4}{4}\log^2 x - \frac{1}{2}\frac{x^4}{4}\log x + \frac{1}{2}\int \frac{\cancel{x^4}^{3}}{4}\cdot\frac{1}{\cancel{x}}\, dx$

$= \frac{x^4}{4}\log^2 x - \frac{x^4}{8}\log x + \frac{x^4}{32} + c = \underline{\frac{x^4}{4}\left(\log^2 x - \frac{\log x}{2} + \frac{1}{8}\right) + c}$

iii) $\int x^3\sqrt{2-x^2}\, dx = \int x^2\left(x\sqrt{2-x^2}\right) dx = -\frac{1}{3}x^2(2-x^2)^{3/2} + \frac{2}{3}\int x(2-x^2)^{3/2} dx$

$= \underline{-\frac{1}{3}x^2(2-x^2)^{3/2} - \frac{2}{15}(2-x^2)^{5/2} + c}.$

iv) $\int x^2 \sin x\, dx = -x^2\cos x + \int 2x\cos x\, dx = -x^2\cos x + 2x\sin x - \int 2\sin x\, dx$

$= -x^2\cos x + 2x\sin x + 2\cos x + c = \underline{(2-x^2)\cos x + 2x\sin x + c}.$

v) $\displaystyle\int \frac{x}{\cos^2 x}\,dx = x\,tg\,x - \int tg\,x\,dx = \underline{x\,tg\,x + \log|\cos x| + c}.$

vi) $\displaystyle\int x^2 e^x\,dx = x^2 e^x - \int 2x e^x\,dx = x^2 e^x - 2x e^x + \int 2e^x + c =$

$$= x^2 e^x - 2x e^x + 2e^x + c = \underline{e^x\left(x^2-2x+2\right) + c}.$$

vii) $\displaystyle\int x\cos^2 x\,dx = \frac{x^2}{2}\cos^2 x + \int \frac{x^2}{2}\, 2\cos x \sin x\,dx = \frac{x^2}{2}\cos^2 x + \frac{1}{2}\int x^2 \sin 2x\,dx$

$$= \frac{x^2}{2}\cos^2 x + \frac{1}{2}\left[-\frac{x^2}{2}\cos 2x + \int x\cos 2x\,dx\right] =$$

$$= \frac{x^2}{2}\cos^2 x - \frac{x^2}{4}\cos 2x + \frac{1}{2}\,\frac{x}{2}\sin 2x - \frac{1}{2}\int \frac{\sin 2x}{2}\,dx =$$

$$= \frac{x^2}{2}\cos^2 x - \frac{x^2}{4}\cos 2x + \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + c$$

$$= \frac{x^2}{2}\cos^2 x - \frac{x^2}{4}\left(\cos^2 x - \sin^2 x\right) + \frac{x}{2}\sin x\cos x + \frac{1}{8}\left(\cos^2 x - \sin^2 x\right) + c$$

$$= \frac{x^2}{2}\cos^2 x - \frac{x^4}{4}\left(2\cos^2 x - 1\right) + \frac{x}{2}\sin x\cos x + \frac{1}{8}\left(2\cos^2 x - 1\right) + c$$

$$= \underline{\frac{1}{4}\left(x^2 + \cos^2 x\right) + \frac{x}{2}\sin x\cos x + c}$$

viii) $\displaystyle\int 3x e^{-x}\,dx = 3\int x e^{-x}\,dx = 3\left[x\frac{e^{-x}}{-1} + \int e^{-x}\,dx\right] = 3\left[-x e^{-x} - e^{-x}\right] + c$

$$= \underline{-3e^{-x}(x+1) + c}.$$

ix) $\displaystyle\int \log(1+2x)\,dx = \int 1\cdot \log(1+2x)\,dx = x\log(1+2x) - \int \frac{2x}{1+2x}\,dx$

definito solo per $1+2x>0$, $x>-\frac{1}{2}$

$$\overset{\text{trucco algebrico}}{=} x\log(1+2x) - \int \frac{\cancel{1+2x}}{\cancel{1+2x}}^{\,1}\,dx + \int \frac{1}{1+2x}\,dx =$$

$$= \underline{x\log(1+2x) - x + \frac{1}{2}\log(1+2x) + c}.$$

e quindi anche qui possiamo tralasciare il $|\cdot|$.

x) $\displaystyle\int 3x\,arctg\,x\,dx = 3\frac{x^2}{2}\,arctg\,x - \frac{3}{2}\int x^2\,\frac{1}{1+x^2}\,dx \overset{\text{trucco algebrico}}{=} \frac{3}{2}x^2\,arctg\,x - \frac{3}{2}\int \frac{1+x^2}{1+x^2}\,dx$

$$+ \frac{3}{2}\int \frac{1}{1+x^2}\,dx = \frac{3}{2}x^2\,arctg\,x - \frac{3}{2}x + \frac{3}{2}\,arctg\,x + c$$

$$= \underline{\frac{3}{2}\left[x^2\,arctg\,x - x + arctg\,x\right] + c}.$$

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xi) $\int \frac{\log(1+2x)}{x^2}\,dx = \int x^{-2}\log(1+2x)\,dx = \frac{x^{-1}}{-1}\log(1+2x) + \int x^{-1}\frac{2}{1+2x}\,dx$

ha solo senso se $1+2x>0$, ossia $x>-\frac{1}{2}$

$$= -\frac{\log(1+2x)}{x} + \int \frac{2}{x(1+2x)}\,dx =$$

$$= -\frac{\log(1+2x)}{x} + 2\left[\int \frac{1}{x}\,dx - \int \frac{2}{1+2x}\,dx\right]$$

$$= -\frac{\log(1+2x)}{x} + 2\log|x| - 2\log(1+2x) + c$$

$$= \underline{-\frac{\log(1+2x)}{x} + 2\log\frac{|x|}{1+2x} + c}.$$

xii) $\int \log x\,(\sin x + x\cos x)\,dx = (\log x)(x\sin x) - \int \frac{1}{\cancel{x}}\cancel{x}\sin x\,dx =$

$$= (\log x)(x\sin x) + \cos x + c.$$

$$= \underline{x\sin x\,\log x + \cos x + c}.$$

xiii) $\int (x+1)^2\cos x\,dx = (x+1)^2\sin x - \int 2(x+1)\sin x\,dx =$

$$= (x+1)^2\sin x - \left[-2(x+1)\cos x + \int 2\cos x\,dx\right]$$

$$= \underline{(x+1)^2\sin x + 2(x+1)\cos x - 2\sin x + c}.$$

xiv) $\int e^x\cos x\,dx = e^x\cos x + \int e^x\sin x\,dx =$

$= e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$. Quindi

$2\int e^x\cos x\,dx = e^x(\cos x + \sin x) + c$, da cui

$$\int e^x\cos x\,dx = \underline{\frac{e^x}{2}(\cos x + \sin x) + c}.$$

5) i) $\int \frac{2x^2-3x+1}{x-4}\,dx =$

$\int (2x+5)\,dx + \int \frac{21}{x-4}\,dx$

$= \underline{x^2 + 5x + 21\log|x-4| + c}.$

$$\begin{array}{l} 2x^2-3x+1 : x-4 = 2x+5 \\ -2x^2+8x \\ \hline /\quad 5x+1 \\ \quad -5x+20 \\ \hline \quad /\quad 21 \end{array}$$

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ii) $\displaystyle\int \frac{2x+1}{x^2-6x+8}\,dx$.

Notiamo che $x^2-6x+8=(x-4)(x-2)$ e quindi

$$\frac{2x+1}{x^2-6x+8}=\frac{a}{x-4}+\frac{b}{x-2}=\frac{a(x-2)+b(x-4)}{(x-4)(x-2)}=$$

$$=\frac{(a+b)x-2a-4b}{(x-4)(x-2)}.$$

Uguagliando i coefficienti dei polinomi al numeratore, si ottiene

$$\begin{cases} a+b=2 \\ -2a-4b=1 \end{cases} \qquad \begin{cases} a=2-b \\ -2(2-b)-4b=1 \end{cases} \qquad \begin{array}{l} a=9/2 \\ b=-5/2 \end{array}$$

Allora

$$\int \frac{2x+1}{x^2-6x+8}\,dx=+\frac{9}{2}\int\frac{1}{x-4}\,dx-\frac{5}{2}\int\frac{1}{x-2}\,dx=$$

$$=+\frac{9}{2}\log|x-4|-\frac{5}{2}\log|x-2|+c .$$

iii) $\displaystyle\int \frac{x^4-3x^3+x+2}{x^2-1}\,dx$

$$\begin{array}{l} x^4-3x^3+x+2 \;:\; x^2-1 = x^2-3x+1 \\ -x^4 \mp x^2 \\ \hline \quad -3x^3+x^2+x+2 \\ \quad \mp 3x^3 \pm 3x \\ \hline \qquad x^2-2x+2 \\ \qquad -x^2 \mp 1 \\ \hline \qquad -2x+3 \end{array}$$

$$=\int(x^2-3x+1)\,dx+\int\frac{-2x+3}{x^2-1}\,dx$$

$$=\frac{x^3}{3}-3\frac{x^2}{2}+x+\int\frac{-2x+3}{x^2-1}\,dx$$

Notiamo che $x^2-1=(x-1)(x+1)$ e quindi

$$\frac{-2x+3}{x^2-1}=\frac{a}{x-1}+\frac{b}{x+1}=\frac{a(x+1)+b(x-1)}{(x-1)(x+1)}$$

$$=\frac{(a+b)x+a-b}{(x-1)(x+1)} .$$

Procedendo come sopra si ha il sistema $\begin{cases} a+b=-2 \\ a-b=3 \end{cases}$, ossia $\begin{cases} a=-2-b \\ -2-b-b=3 \end{cases}$

da cui $\begin{cases} a=1/2 \\ b=-5/2 \end{cases}$. Allora

$$\int\frac{x^4-3x^3+x+2}{x^2-1}\,dx=\frac{x^3}{3}-\frac{3x^2}{2}+x+\frac{1}{2}\log|x-1|-\frac{5}{2}\log|x+1|+c .$$

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iv) $\displaystyle\int \frac{x^3-2x^2-x+3}{x^2+x+1}\,dx$

$$\begin{array}{l} x^3-2x^2-x+3 \;:\; x^2+x+1 = x-3 \\ \underline{-x^3 \pm x^2 \pm x} \\ /\;\; -3x^2-2x+3 \\ \underline{\mp 3x^2 \mp 3x \mp 3} \\ /\;\;\; x+6 \end{array}$$

$$= \int (x-3)\,dx + \int \frac{x+6}{x^2+x+1}\,dx$$

$$= \frac{x^2}{2} - 3x + \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{11}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$= \frac{x^2}{2} - 3x + \frac{1}{2}\log(x^2+x+1) + \frac{11}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$$

Poiché $\displaystyle\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{1}{3/4}\int \frac{1}{\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)^2+1}\,dx$

$$= \frac{2}{\sqrt{3}}\int \frac{2/\sqrt{3}}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,dx = \frac{2}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)+c$$

si ha

$$\int \frac{x^3-2x^2-x+3}{x^2+x+1}\,dx = \underline{\underline{\frac{x^2}{2} - 3x + \frac{1}{2}\log(x^2+x+1) + \frac{11}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)+c}}\,.$$

v) $\displaystyle\int \frac{x+2}{(x+1)^2}\,dx \underset{\substack{x+1=t \\ x=t-1 \\ dx=dt}}{=} \int \frac{t+1}{t^2}\,dt = \int\left(\frac{1}{t}+\frac{1}{t^2}\right)dt = \log|t| - \frac{1}{t} + c$

$$= \underline{\underline{\log|1+x| - \frac{1}{x+1} + c}}\,.$$

vi) $\displaystyle\int \frac{x^2}{x^2-3x-4}\,dx =$

$$\begin{array}{l} x^2 \;:\; x^2-3x-4 = 1 \\ \underline{-x^2 \pm 3x \pm 4} \\ /\;\; 3x+4 \end{array}$$

$$= \int 1\,dx + \int \frac{3x+4}{x^2-3x-4}\,dx$$

Notiamo che $x^2-3x-4=(x-4)(x+1)$ e quindi

$$\frac{3x+4}{x^2-3x-4} = \frac{a}{x-4} + \frac{b}{x+1} = \frac{a(x+1)+b(x-4)}{(x-4)(x+1)}$$

$$= \frac{(a+b)x + a - 4b}{(x-4)(x+1)}$$

Segue $\begin{cases} a+b=3 \\ a-4b=4 \end{cases} \iff \begin{cases} a=3-b \\ 3-b-4b=4 \end{cases}$ $\quad a=\frac{16}{5}$, $b=-\frac{1}{5}$

Quindi

$$\int \frac{x^2}{x^2-3x-4}\,dx = \underline{\underline{x + \frac{16}{5}\log|x-4| - \frac{1}{5}\log|x+1| + c}}.$$

6) i) $\int \cos\sqrt{x}\,dx \underset{\uparrow}{=} \int 2t\cos t\,dt = 2t\,\mathrm{sen}t - \int 2\,\mathrm{sen}t\,dt =$

$\sqrt{x}=t$

$x=t^2$ $\quad = 2t\sin t + 2\cos t + c$

$dx = 2t\,dt$ $\quad = \underline{\underline{2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + c}}$

ii) $\int \frac{\log(\log x)}{x}\,dx \underset{\uparrow}{=} \int \frac{\log t}{\cancel{e^t}}\cdot\cancel{e^t}\,dt = t\log t - \int \cancel{t}\cdot\frac{1}{\cancel{t}}\,dt$

$\log x = t$

$x=e^t$ $\quad = t(\log t - 1) + c$

$dx = e^t\,dt$ $\quad = \underline{\underline{\log x\left(\log(\log x) - 1\right) + c}}.$

iii) $\int \frac{1}{\sin x + 1}\,dx \underset{\uparrow}{=} \int \frac{1}{\frac{2t}{1+t^2}+1}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{t^2+2t+1}\,dt =$

$\sin x = \frac{2t}{1+t^2}$ $\quad = 2\int \frac{1}{(t+1)^2}\,dt = -2(t+1)^{-1} + c$

$dx = \frac{2}{1+t^2}\,dt$

$t = \mathrm{tg}\frac{x}{2}$ $\quad = \underline{\underline{-\frac{2}{\left(\mathrm{tg}\frac{x}{2}+1\right)} + c}}$

iv) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \underset{\uparrow}{=} \int \frac{6t^5}{t^3+t^2}\,dt = \int \frac{6t^3}{t+1}\,dt$

$\sqrt[6]{x} = t$

$x = t^6$ $\quad = 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right)dt$

$dx = 6t^5\,dt$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] + c$$

$$= \underline{\underline{2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\log|\sqrt[6]{x}+1| + c}}.$$

7) a) $f(x) = \frac{\log(x+3)}{x^2}$; supponiamo $x>0$

Cerchiamo l'insieme delle primitive di f : $\int f(x)\,dx = \int \frac{\log(x+3)}{x^2}\,dx =$

$$= -\frac{1}{x}\log(x+3) + \int \frac{1}{x}\cdot\frac{1}{x+3}\,dx = -\frac{1}{x}\log(x+3) + \frac{1}{3}\int\left(\frac{1}{x} - \frac{1}{x+3}\right)dx =$$

$$= -\frac{1}{x}\log(x+3) + \frac{1}{3}\log x - \frac{1}{3}\log(x+3) + c$$

Imponendo che $P=(1,0)$ appartenga al grafico di $F_c(x) = -\frac{1}{x}\log(x+3) + \frac{1}{3}\log x - \frac{1}{3}\log(x+3) + c$ si ottiene

$$0 = -\log 4 - \frac{1}{3}\log 4 + c$$

ossia $\boxed{c = \frac{4}{3}\log 4}$.

□

b) $g(x) = \sqrt{1+\sqrt{x}}$; supponiamo $x \geq 0$.

Cerchiamo l'insieme delle primitive di g:

$$\int g(x)\,dx = \int \sqrt{1+\sqrt{x}}\,dx =$$

$t = \sqrt{1+\sqrt{x}}$ si ha $\sqrt{x} = t^2-1$, $x = (t^2-1)^2$, $dx = 2(t^2-1)2t$

$$= \int t \cdot 4t(t^2-1)\,dt = \int (4t^4 - 4t^2)\,dt = 4\left[\frac{t^5}{5} - \frac{t^3}{3}\right] + c$$

$$= 4\left(\frac{\sqrt{(1+\sqrt{x})^5}}{5} - \frac{\sqrt{(1+\sqrt{x})^3}}{3}\right) + c.$$

Imponendo che $P=(0,0)$ appartenga al grafico di $G_c = 4\left(\frac{\sqrt{(1+\sqrt{x})^5}}{5} - \frac{\sqrt{(1+\sqrt{x})^3}}{3}\right) + c$

si ottiene $\quad 0 = 4\left(\frac{1}{5} - \frac{1}{3}\right) + c \quad \Rightarrow \boxed{c = \frac{8}{15}}$

□

c) $h(x) = x\log(x^2+1)$;

L'insieme delle primitive di $h(x)$ è dato da

$$\int h(x)\,dx = \int x\log(x^2+1)\,dx = \frac{x^2}{2}\log(x^2+1) - \int \frac{x^2}{\cancel{2}} \cdot \frac{1}{x^2+1} \cdot \cancel{2}x\,dx$$

$$= \frac{x^2}{2}\log(x^2+1) - \int \frac{x^3}{x^2+1}\,dx =$$

$$= \frac{x^2}{2}\log(x^2+1) - \int\left(x - \frac{x}{x^2+1}\right)dx$$

$x^3 : x^2+1 = x$

$-x^3 \pm x$

$/ \; -x$

$$= \frac{x^2}{2}\log(x^2+1) - \frac{x^2}{2} + \frac{1}{2}\log(x^2+1) + c$$

Procedendo come sopra abbiamo $\log 2 = \cancel{\frac{\log 2}{2}} - \frac{1}{2} + \cancel{\frac{\log 2}{2}} + c$,

ossia $\boxed{c = \frac{1}{2}}$.

■

E9

8) i) $\displaystyle\int_0^1 \frac{x-1}{x^2-4}\,dx =$

$$\frac{x-1}{(x-2)(x+2)} = \frac{a}{x-2} + \frac{b}{x+2} = \frac{a(x+2)+b(x-2)}{(x-2)(x+2)}$$

$$= \frac{(a+b)x+2a-2b}{(x-2)(x+2)} \Leftrightarrow$$

$$\begin{cases} a+b=1 \\ 2a-2b=-1 \end{cases} \Leftrightarrow \begin{matrix} a=1-b \\ 2-2b-2b=-1 \end{matrix} \quad \begin{cases} a=1/4 \\ b=3/4 \end{cases}$$

$$= \frac{1}{4}\int_0^1 \frac{1}{x-2}\,dx + \frac{3}{4}\int_0^1 \frac{1}{x+2}\,dx$$

$$= \frac{1}{4}\Big[\log|x-2|\Big]_0^1 + \frac{3}{4}\Big[\log|x+2|\Big]_0^1$$

$$= \frac{1}{4}\Big[\cancel{\log 1} - \log 2\Big] + \frac{3}{4}\Big[\log 3 - \log 2\Big] = \underline{\underline{\frac{3}{4}\log 3 - \log 2.}}$$

ii) $\displaystyle\int_9^{16} \frac{\sqrt{x}-3}{x-3\sqrt{x}+2}\,dx =$

$$\int \frac{\sqrt{x}-3}{x-3\sqrt{x}+2}\,dx \underset{\uparrow}{=} \int \frac{t-3}{t^2-3t+2}\cdot 2t\,dt$$

$\sqrt{x}=t$
$x=t^2$
$dx=2t\,dt$

$$= 2\int \frac{t^2-3t}{t^2-3t+2}\,dt = 2\int \frac{t^2-3t+2-2}{t^2-3t+2}\,dt =$$

$$= 2\int dt - 4\int \frac{1}{(t-2)(t-1)}\,dt =$$

$$= 2t - 4\int \left[\frac{1}{t-2} - \frac{1}{t-1}\right] dt =$$

$$= 2t - 4\Big[\log|t-2| - \log|t-1|\Big] + c =$$

$$= 2\sqrt{x} - 4\Big[\log|\sqrt{x}-2| - \log|\sqrt{x}-1|\Big] + c$$

$$= \boxed{\Big[2\sqrt{x} - 4\big[\log|\sqrt{x}-2| - \log|\sqrt{x}-1|\big]\Big]_9^{16}}$$

$$= \Big(2\cdot 4 - 4\big[\log 2 - \log 3\big]\Big) -$$

$$- \Big(2\cdot 3 - 4\big[\cancel{\log 1} - \log 2\big]\Big)$$

$$= 8 - 8\log 2 + 4\log 3 - 6$$

$$= \underline{\underline{2 - 8\log 2 + 4\log 3.}}$$

iii) $\displaystyle\int_0^1 x \sin(x^2-5)\,dx = \left[-\frac{\cos(x^2-5)}{2}\right]_0^1$

$$= -\frac{\cos(-4)}{2} + \frac{\cos(-5)}{2} = \underline{\underline{\frac{\cos 5 - \cos 4}{2}}}.$$

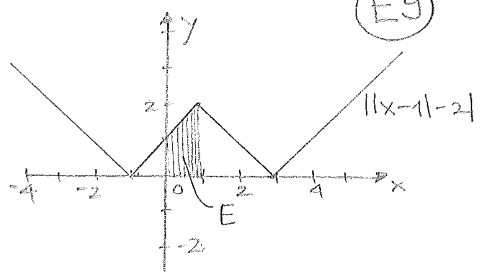
iv) $\displaystyle\int_1^2 \frac{x+2\sqrt[3]{x}}{x^2}\,dx = \int_1^2 \left(\frac{1}{x} + 2x^{\frac{1}{3}-2}\right)dx = \int_1^2 \left(\frac{1}{x} + 2x^{-5/3}\right)dx =$

$$= \left[\log x + \frac{2x^{-2/3}}{-2/3}\right]_1^2 = \left(\log 2 - 3\cdot 2^{-2/3}\right) - \left(\cancel{\log 1} - 3\right)$$

$$= \underline{\underline{\log 2 + 3 - \frac{3}{\sqrt[3]{4}}}}.$$

E9

v) $\displaystyle\int_0^1 \big||x-1|-2\big|\,dx = \text{area } E = \underline{\underline{\tfrac{3}{2}}}$

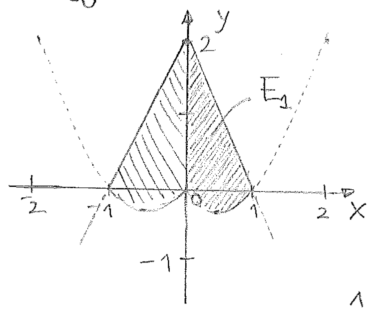


vi) $\displaystyle\int_{-2}^{1} e^{-|x|}\,dx = \int_{-2}^{0} e^{x}\,dx + \int_0^1 e^{-x}\,dx$

$$= \Big[e^x\Big]_{-2}^{0} - \Big[e^{-x}\Big]_0^1 = (1-e^{-2}) - (e^{-1}-1) = \underline{\underline{2-\frac{1}{e}-\frac{1}{e^2}}}\,.$$

9) i) $f(x) = x^2 - |x|$

$g(x) = -2|x| + 2$

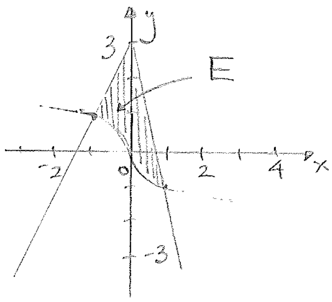


area $E = 2$ area E_1

$$= 2\int_0^1 \Big[(-2x+2) - (x^2-x)\Big]dx = 2\int_0^1 (-x^2-x+2)\,dx =$$

$$= 2\Big[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\Big]_0^1 = 2\Big[-\frac{1}{3}-\frac{1}{2}+2\Big] = \cancel{2}\cdot\frac{7}{\cancel{6}_3} = \underline{\underline{\frac{7}{3}}}\,.$$

ii) $f(x) = -\sqrt[3]{x} \qquad y = 2x+3 \qquad y = -4x+3$



$$\text{area } E = \int_{-1}^{0}\big(2x+3-(-\sqrt[3]{x})\big)dx + \int_0^1\big(-4x+3-(-\sqrt[3]{x})\big)dx$$

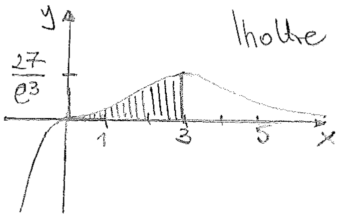
$$= \int_{-1}^{0}(2x+3+\sqrt[3]{x})\,dx + \int_0^1(-4x+3+\sqrt[3]{x})\,dx$$

$$= \Big[x^2+3x+\frac{x^{4/3}}{4/3}\Big]_{-1}^{0} + \Big[-2x^2+3x+\frac{x^{4/3}}{4/3}\Big]_0^1$$

$$= -\Big(1-3+\frac{3}{4}\Big) + \Big(-2+3+\frac{3}{4}\Big)$$

$$= \cancel{2}-\cancel{\tfrac{3}{4}}\ \cancel{-2}+3+\cancel{\tfrac{3}{4}} = \underline{\underline{3}}\,.$$

iii) $f(x) = x^3 e^{-x}$; facendo un breve studio di funzioni si deduce il seg. grafico:



Inoltre $y=0$ retta tg al grafico di f in $(0,0)$. Risulta

$$\text{area } E = \int_0^3 f(x)\,dx = \int_0^3 x^3e^{-x}\,dx \underset{\substack{\uparrow\\ \text{integrando per parti}}}{=} \Big[-e^{-x}(x^3+3x^2+6x+6)\Big]_0^3 =$$

$$= -e^{-3}(27+27+18+6) + 6 = 6 - \frac{78}{e^3} \sim 2.1$$