Proof. From $\psi(x) = \psi(x) - \psi(x_1) = \int_{x_1}^{x} \psi'(t)\,dt$, $x \in I$, we infer that

$$|\psi(x)|^2 \le \left(\int_{x_1}^{x_2} |\psi'|\,dt\right)^2 \le (x_2 - x_1)\int_{x_1}^{x_2} |\psi'|^2\,dt$$

holds, taking Schwarz's inequality into account. Integrating with respect to x from x_1 to x_2, the desired inequality follows at once. □

Now we conclude:

Theorem. *If $a > 0$ holds on I, and if there exists a Jacobi field $v \in C^2(I)$ with $v > 0$ on I, then u is a strict weak minimizer of $\mathscr{F}$.*

Proof. Let φ be an arbitrary function in $C_0^1(I)$. Then also $\psi := \varphi/v \in C_0^1(I)$, and we infer from Jacobi's lemma and from Poincaré's inequality that

$$\begin{aligned}\mathscr{Q}(\varphi) &= \int_{x_1}^{x_2} av^2\psi'^2\,dx \ge \inf_I\{av^2\}\int_{x_1}^{x_2}\psi'^2\,dx \\ &\ge \inf_I\{av^2\}\frac{1}{|x_2 - x_1|^2}\int_{x_1}^{x_2}\psi^2\,dx \\ &\ge \inf_I\{av^2\}\inf_I\left\{\frac{1}{v^2}\right\}\frac{1}{|x_2 - x_1|^2}\int_{x_1}^{x_2}\varphi^2\,dx.\end{aligned}$$

In other words, there is a constant $\mu > 0$ such that

$$\mathscr{Q}(\varphi) \ge \mu\int_{x_1}^{x_2}\varphi^2\,dx \tag{9}$$

holds for all $\varphi \in C_0^1(I)$.

We now claim that there is some $\lambda > 0$ such that

$$\mathscr{Q}(\varphi) \ge \lambda\int_{x_1}^{x_2}(\varphi'^2 + \varphi^2)\,dx \tag{10}$$

is satisfied for all $\varphi \in C_0^1(I)$; then Proposition 2 yields that u is a weak minimizer.

In order to prove (10), we recall that

$$\mathscr{Q}(\varphi) = \int_{x_1}^{x_2}(a\varphi'^2 + 2b\varphi\varphi' + c\varphi^2)\,dx.$$

Setting

$$\alpha := \inf_I a, \qquad \beta := \sup_I |b|, \qquad \gamma := \sup_I |c|,$$

we obtain that

$$\alpha\int_{x_1}^{x_2}\varphi'^2\,dx \le \mathscr{Q}(\varphi) + 2\beta\int_{x_1}^{x_2}|\varphi||\varphi'|\,dx + \gamma\int_{x_1}^{x_2}\varphi^2\,dx. \tag{11}$$

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Now we apply the inequality

$$2|xy| \le \varepsilon x^2 + \varepsilon^{-1}y^2, \quad \varepsilon > 0,$$

to the second integral of (11); it follows that

$$\alpha\int_{x_1}^{x_2}\varphi'^2\,dx \le \mathscr{Q}(\varphi) + \varepsilon\int_{x_1}^{x_2}\varphi'^2\,dx + (\varepsilon^{-1}\beta^2 + \gamma)\int_{x_1}^{x_2}\varphi^2\,dx.$$

Choosing $\varepsilon = \alpha/2$ and taking (9) into account, we arrive at

$$\begin{aligned}\frac{\alpha}{2}\int_{x_1}^{x_2}\varphi'^2\,dx &\le \mathscr{Q}(\varphi) + (2\alpha^{-1}\beta^2 + \gamma)\int_{x_1}^{x_2}\varphi^2\,dx \\ &\le [1 + \mu^{-1}(2\alpha^{-1}\beta + \gamma)]\mathscr{Q}(\varphi).\end{aligned} \tag{12}$$

Then (10) follows from (9) and (12). □

Remark. We shall see below that the converse of Theorem 1 holds true in the sense that the existence of a "nonvanishing" Jacobi field along u is a necessary condition for u to be a weak minimizer. More precisely, we have: *If $a > 0$ on I, and if u is a weak minimizer of $\mathscr{F}$, then for any $x_0 \in [x_1, x_2]$ the solution v of the Cauchy problem*

$$\mathscr{J}_u v = 0, \qquad v(x_0) = 0, \qquad v'(x_0) = 1$$

has no zeros different from x_0 in (x_1, x_2). This fact is often referred to as *Hilbert's necessary condition.*

2.2. Jacobi Fields and Conjugate Values

In order to gain more information about Jacobi fields, we note that Jacobi's equation

$$-(av')' + (c - b')v = 0$$

is equivalent to the equation

$$v'' + pv' + qv = 0, \tag{1}$$

with

$$p := \frac{a'}{a}, \qquad q := \frac{b' - c}{a} \tag{1'}$$

if we assume that $a > 0$ holds on I. For the sake of simplicity, *we will in the following suppose that $a > 0$ is true on $I_0 := (x_1 - \delta_0, x_2 + \delta_0)$, $\delta_0 > 0$.*

We use the fact that, for every $x_0 \in I$ and arbitrary numbers $\alpha, \beta \in \mathbb{R}$, the initial value problem

$$v(x_0) = \alpha, \qquad v'(x_0) = \beta \tag{2}$$

for (1) has exactly one C^2-solution on I which will be denoted by $\omega(x; x_0, \alpha, \beta)$, and we can assume that ω is a C^2-solution of (1) on the larger interval $I_0 = (x_1 - \delta_0, x_2 + \delta_0)$.

Fix some $x_0 \in I$ and set

$$(3') \qquad v_1(x) := \omega(x; x_0, 1, 0), \qquad v_2(x) := \omega(x; x_0, 0, 1).$$

Then $\omega(x; x_0, \alpha, \beta)$ describes all solutions of (1), and we can write

$$(3'') \qquad \omega(x; x_0, \alpha, \beta) = \alpha v_1(x) + \beta v_2(x),$$

taking the uniqueness theorem into account. Hence the set of solutions of (1) forms a two-dimensional linear space X over $\mathbb{R}$. Every pair $\{v_1, v_2\}$ of linearly independent vectors $v_1, v_2 \in X$ is called a *fundamental system* or *base* of X.[5]

Proposition 1. *Let $v \in X$, and suppose that $v(x) \not\equiv 0$. Then the following holds:*

(i) If $x_0 \in I_0$ and $v(x_0) = 0$, then $v'(x_0) \neq 0$.

(ii) The zeros of v are isolated in I_0.

(iii) If $x_0 \in I_0$, $v(x_0) = 0$, and $\tilde{v} \in X$, then v, $\tilde{v}$ are linearly dependent if and only if $\tilde{v}(x_0) = 0$.

Proof. (i) follows from the uniqueness theorem and implies (ii); (iii) is a consequence of the two formulas (3′) and (3″). □

Proposition 2. *Suppose that $v_1, v_2 \in X$, and let*

$$W := \begin{vmatrix} v_1 & v_2 \\ v_1' & v_2' \end{vmatrix} = v_1 v_2' - v_2 v_1'$$

be the Wronskian of the pair $\{v_1, v_2\}$. Then we have:

(i) $\{v_1, v_2\}$ is a base of X exactly if $W(x) \neq 0$.

(ii) For every $x_0 \in I_0$, we have

$$(4) \qquad W(x) = W(x_0) a(x_0) \frac{1}{a(x)} \quad \textit{for all } x \in I_0 .$$

(iii) If $\{v_1, v_2\}$ is a base of X, then between two neighbouring zeros of v_1 there lies exactly one zero of v_2, and vice versa.

Proof. (i) follows immediately from the uniqueness theorem.

(ii) A brief computation yields $W' + pW = 0$ and $p = a'/a$, whence $0 = a'W + aW' = (aW)'$; thus $aW = \text{const}$. The last relation implies (4).

(iii) Since $\{v_1, v_2\}$ is a base of X, we have $W(x) \neq 0$, and we may assume that $W(x) > 0$. Let ξ, ξ^* be two neighbouring zeros of v_1, i.e., $v_1(x) \neq 0$ for $\xi < x < \xi^*$.

[5] For general results about the theory of ordinary differential equations see e.g. Hartman [1] or Coddington–Levinson [1].

Since $v_1'(\xi) \neq 0$ and $v_1'(\xi^*) \neq 0$, it follows that $v_1'(\xi)$ and $v_1'(\xi^*)$ have opposite signs. From $W(x) > 0$ we infer $-v_1'(\xi)v_2(\xi) > 0$ and $-v_1'(\xi^*)v_2(\xi^*) > 0$, and we arrive at $v_2(\xi) > 0$, $v_2(\xi^*) < 0$ or, vice versa, $v_2(\xi) < 0$, $v_2(\xi^*) > 0$. Thus there exists a zero of v_2 between ξ and ξ^*. If there were a second zero of v_2 in (ξ, ξ^*), then by the same reasoning we could deduce the existence of another zero of v_1 between ξ and ξ^* which is impossible. This proves one implication of (iii), and the other is obvious. □

Property (iii) of the Proposition 2 is known as *Sturm's oscillation theorem.*

For some fixed parameter values $\xi \in I_0$, we now introduce *Jacobi's function*

$$(5) \qquad \Delta(x, \xi) := \omega(x; \xi, 0, 1).$$

In other words, $v(x) = \Delta(x, \xi)$ is the uniquely determined Jacobi field on I_0 with $v(\xi) = 0$ and $v'(\xi) = 1$.

Definition. *The (isolated) zeros of $\Delta(\cdot, \xi)$ will be called* conjugate values *to ξ, irrespectively of whether they lie to the left or to the right of ξ.*

Obviously ξ is conjugate to ξ^* exactly if ξ^* is conjugate to ξ. The smallest conjugate value ξ^* with $\xi^* > \xi$ is called the *next conjugate value to* ξ (to the right). If ξ, ξ^* is a pair of conjugate values, then the points $P = (\xi, u(\xi))$ and $P^* = (\xi^*, u(\xi^*))$ will be called a *pair of conjugate points* on the extremal $z = u(x)$.

Proposition 3. *If $(x_1, x_2]$ does not contain any conjugate point to x_1, then there is no pair ξ, ξ^* of conjugate points contained in I.*

Proof. Suppose that ξ, $\xi^* \in I$ are conjugate to each other. By assumption, we have $x_1 < \xi < \xi^* \le x_2$. Set $v_1(x) := \Delta(x, x_1)$ and $v_2(x) := \Delta(x, \xi)$. From Proposition 1(iii) we infer that v_1, v_2 are linearly independent whence, by Sturm's theorem, v_1 must have a zero between ξ and ξ^*, which, by assumption, is impossible. □ da cui

Recall that we are actually interested in the piece $z = u(x)$, $x_1 \le x \le x_2$, of the extremal u. We distinguish three possibilities for $I = [x_1, x_2]$:

Case 1: There exists no conjugate value to x_1 in I.

Case 2: There is a conjugate value to x_1 in int $I = (x_1, x_2)$.

Case 3: The value x_2 is the next conjugate value to x_1 in I.

Corresponding to these three cases, we can now state necessary as well as sufficient conditions for u to be a weak minimizer.

Theorem 1. *Suppose that the strict Legendre condition*

$$a(x) = \tfrac{1}{2} F_{pp}(x, u(x), u'(x)) > 0$$

is fulfilled on I.

(i) If there exists no conjugate value to x_1 in I, then u is a strict weak minimizer.

(ii) If there is a conjugate value to x_1 in $(x_1, x_2) = \text{int}\, I$, then u is not a weak minimizer.

(iii) If the value x_2 is the next conjugate value to x_1 in I, then it cannot be decided without a more penetrating discussion whether u is a weak minimizer or not.

Proof. Without loss of generality we can assume that $a(x) > 0$ holds on the larger interval $I_0 = (x_1 - \delta_0, x_2 + \delta_0)$, $0 < \delta_0 \ll 1$.

(i) Suppose now that we are in case 1. Then, by assumption, $\Delta(x, x_1) > 0$ for all $x \in (x_1, x_2]$. On account of Proposition 1(iii), the two Jacobi fields $\Delta(x, x_2)$ and $\Delta(x, x_1)$ have to be independent since $\Delta(x_2, x_2) = 0$.

We claim that there exists some $x_0 \in (x_1 - \delta_0, x_1)$ such that $\Delta(x, x_2) \neq 0$ holds on $[x_0, x_2)$. If not, then there would be some conjugate value to x_2 in $[x_1, x_2)$; let x_2^* be the largest of these. By Sturm's reasoning, $\Delta(x, x_1)$ must have a zero in (x_2^*, x_2) which by assumption is impossible.

With such a value x_0, we form $\Delta(x, x_0)$. Since $\Delta(x_0, x_0) = 0$ but $\Delta(x_0, x_2) \neq 0$, the two Jacobi fields $\Delta(x, x_0)$ and $\Delta(x, x_2)$ are linearly independent. Then it follows that $\Delta(x, x_0) \neq 0$ in $(x_0, x_2]$ (in fact, $\Delta(x, x_0) > 0$). Otherwise Sturm's reasoning would imply that $\Delta(x, x_2)$ possessed a zero in (x_0, x_2), and this is impossible as was shown before.

Hence we have shown that, in case 1, $v(x) := \Delta(x, x_0)$ forms a strictly positive Jacobi field on $I = [x_1, x_2]$. Now we may infer from the theorem of *2.1* that u is a strict weak minimizer of $\mathscr{F}$.

(ii) The assertion for case 2 follows from the next theorem, and in case 3 nothing is to be proved as one can find examples for both situations.[6] $\square$

Theorem 2. *Suppose that the strict Legendre-condition $a(x) > 0$ on I holds, and let v be a Jacobi field, $v(x) \not\equiv 0$, that vanishes at the endpoints of some proper closed subinterval I' of I. Then there exists functions $\varphi \in C_0^1(I)$ such that $\mathscr{Q}(\varphi) < 0$, and consequently u is not a weak minimizer of $\mathscr{F}$.*

Proof. We restrict ourselves to the case where I and I' have the same left endpoint; the general case is handled in the same way.

Let $I' = [x_1, x_1']$, $x_1 < x_1' < x_2$. Since $v \neq 0$ and $v(x_1) = v(x_1') = 0$, the point x_1' is a conjugate value to x_1. We consider the next conjugate value x_1^* to x_1, i.e., $x_1 < x_1^* \le x_1'$, and choose some $\beta \in (x_1^*, x_2)$ such that the interval $(x_1^*, \beta]$ contains no value conjugate to x_1. Then we introduce the two nontrivial Jacobi fields

$$v_1(x) := \Delta(x, x_1), \qquad v_2(x) := -\Delta(x, \beta).$$

Because of $v_1(\beta) \neq 0$ and $v_2(\beta) = 0$, the functions v_1 and v_2 are linearly independent whence, by Sturm's theorem, v_2 possesses exactly one zero α between

[6] Cf. Carathéodory [10], p. 295, and Bolza [3], §47, pp. 357–364.

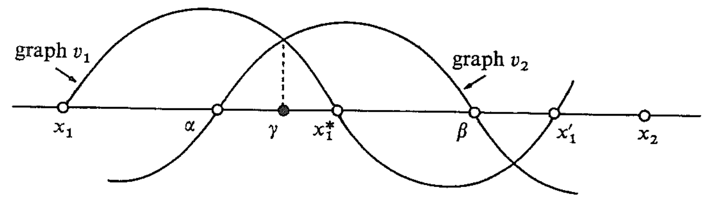

Fig. 2. The functions v_1 and v_2 in Theorem 2.

x_1 and x_1^*, and $v_2(x_1^*) \neq 0$. Moreover, $v_2(x)$ cannot vanish for any $x \in (x_1^*, \beta)$. Otherwise, by Sturm's reasoning, v_1 had to vanish for some value in (x_1^*, β), and this would contradict our choice of β. Hence α is the only zero of v_2 in $[x_1, \beta)$, and $v_2'(\beta) = -1$ implies $v_2(x) > 0$ in (α, β), and therefore $v_2(x) < 0$ in $[x_1, \alpha)$.

Now we consider the Wronskian $W = v_1 v_2' - v_2 v_1'$ which, according to (4), satisfies

$$a(x) W(x) \equiv C \quad \text{on } I,$$

with a constant $C \neq 0$ which can be computed from

$$C = a(x_1) W(x_1).$$

Since $a(x_1) > 0$ and $W(x_1) = -v_2(x_1) > 0$, we infer $C > 0$.

Next we observe that also $v_1, v_2 - v_1$ are linearly independent, and Sturm's theorem yields as well the existence of a zero γ of $v_2 - v_1$ between x_1 and x_1^*, whence $v_1(\gamma) = v_2(\gamma)$ and $\alpha < \gamma < x_1^*$.

Finally we define the test function η by

$$\eta(x) := \begin{cases} v_1(x) & \text{on } [x_1, \gamma], \\ v_2(x) & \text{on } [\gamma, \beta], \\ 0 & \text{on } [\beta, x_2]. \end{cases}$$

Note that η is continuous on I, piecewise of class C^2 and $\eta(x_1) = \eta(x_2) = 0$. Moreover, η' and η'' are only at $x = \gamma$ and $x = \beta$ discontinuous.

Since $Q(x, z, p)$ is a quadratic form with respect to z, p, we have

$$2Q(x, z, p) = zQ_z(x, z, p) + pQ_p(x, z, p)$$

and therefore

$$Q(x, \eta, \eta') = \tfrac{1}{2}\delta Q(\eta, \eta).$$

It follows that

$$\mathscr{Q}(\eta) = \int_{x_1}^{x_2} Q(x, \eta, \eta')\, dx$$
$$= \frac{1}{2}\int_{x_1}^{\gamma} \delta Q(v_1, v_1)\, dx + \frac{1}{2}\int_{\gamma}^{\beta} \delta Q(v_2, v_2)\, dx.$$

Since $v_1(x)$ and $v_2(x)$ are of class C^2 on $[x_1, \gamma]$ and $[\gamma, \beta]$, respectively, we infer

$$\int_{x_1}^{\gamma} \delta Q(v_1, v_1)\, dx = \int_{x_1}^{\gamma} v_1 L_Q(v_1)\, dx + [v_1 Q_p(x, v_1, v_1')]_{x_1}^{\gamma}$$

$$\int_{\gamma}^{\beta} \delta Q(v_2, v_2)\, dx = \int_{\gamma}^{\beta} v_2 L_Q(v_2)\, dx + [v_2 Q_p(x, v_2, v_2')]_{\gamma}^{\beta}.$$

Moreover,

$$Q(x, z, p) = a(x)p^2 + 2b(x)pz + c(x)z^2$$

implies

$$\tfrac{1}{2} Q_p(x, z, p) = a(x)p + b(x)z.$$

Since $v_1(x_1) = 0$, $v_1(\gamma) = v_2(\gamma)$, $v_2(\beta) = 0$, and

$$L_Q(v_1) = \mathscr{J}_u v_1 = 0, \qquad L_Q(v_2) = \mathscr{J}_u v_2 = 0,$$

we obtain

$$\begin{aligned}\mathscr{Q}(\eta) &= a(\gamma)\{v_1(\gamma)v_1'(\gamma) - v_2(\gamma)v_2'(\gamma)\}\\ &= a(\gamma)\{v_2(\gamma)v_1'(\gamma) - v_1(\gamma)v_2'(\gamma)\}\\ &= -a(\gamma)W(\gamma) = -C < 0.\end{aligned}$$

By "smoothing the corners" of η at $x = \gamma$ and $x = \beta$, we can construct a function $\varphi \in C_0^1(I)$ such that also

$$\mathscr{Q}(\varphi) < 0$$

holds; in fact, we can even find some $\varphi \in C_c^\infty(I)$ with $\mathscr{Q}(\varphi) < 0$. Therefore u cannot be a weak minimizer. □

Hilbert's necessary condition formulated in the remark following the theorem in *2.1* is now an immediate consequence of Theorem 1 or Theorem 2.

2.3. Geometric Interpretation of Conjugate Points

Now we turn to a *geometric interpretation of conjugate points* due to Jacobi.

Suppose that the extremal $u(x)$, $x_1 \le x \le x_2$, is embedded in some *one-parameter family of extremals* $\varphi(x, \alpha)$ such that $\varphi(x, \alpha_0) = u(x)$ for $x \in I$. Let φ be of class C^3 on $I \times (\alpha_0 - \delta, \alpha_0 + \delta)$, $\delta > 0$, and assume that $\varphi_{\alpha x}(x, \alpha_0)$ is not identically zero on I. Then $v(x) := \varphi_\alpha(x, \alpha_0)$ is a nontrivial Jacobi field along $u(x)$ as we have stated in the Corollary of *1.2*.

Now we consider the particular case that, for $x = x_1$, all extremals of the family pass through the same point $P_1 = (x_1, z_1)$, i.e.,

$$\varphi(x_1, \alpha) = z_1 \quad \text{for } \alpha_0 - \delta < \alpha < \alpha_0 + \delta.$$

Then we have $v(x_1) = 0$. Suppose that there is a next conjugate point $P_1^*(\alpha_0) = (x_1^*, z_1^*)$, $z_1^* = u(x_1^*)$, $x_1 < x_1^* < x_2$, to P_1 on the extremal $u(x) = \varphi(x, \alpha_0)$. Since $v(x_1) = 0$, it follows that $v(x_1^*) = 0$, and therefore also $v'(x_1) \ne 0$ and $v'(x_1^*) \ne 0$; that is,

$$\varphi_{\alpha x}(x_1, \alpha_0) \ne 0 \quad \text{and} \quad \varphi_{\alpha x}(x_1^*, \alpha_0) \ne 0.$$

Consider now the system of equations

$$(1) \qquad \varphi(x, \alpha) - z = 0, \qquad \varphi_\alpha(x, \alpha) = 0$$

for the unknown variables x, z, α which has the solution x_1^*, z_1^*, α_0. Because of $\varphi_{\alpha x}(x_1^*, \alpha_0) \ne 0$, we can apply the implicit function theorem and obtain a C^2-function $x = \xi(\alpha)$ with $x_1^* = \xi(\alpha_0)$ such that $\varphi_\alpha(\xi(\alpha), \alpha) = 0$ holds. If we set $\zeta(\alpha) := \varphi(\xi(\alpha), \alpha)$, then $(x, z, \alpha) = (\xi(\alpha), \zeta(\alpha), \alpha)$, $\alpha \in \mathscr{U}(\alpha_0)$, is a parameterization of the set of solutions of (1). If we assume that

$$\varphi_{\alpha\alpha}(x_1^*, \alpha_0) \ne 0,$$

then it follows from

$$\varphi_{\alpha\alpha}(\xi(\alpha), \alpha) + \varphi_{\alpha x}(\xi(\alpha), \alpha)\xi_\alpha(\alpha) = 0$$

that $\xi_\alpha(\alpha) \ne 0$ holds in some neighborhood $\mathscr{U}(\alpha_0)$ of α_0, and therefore

$$\mathscr{E} = \{P^*(\alpha) \colon P^*(\alpha) = (\xi(\alpha), \zeta(\alpha)), \alpha \in \mathscr{U}(\alpha_0)\}$$

describes a regular curve of class C^2. It is part of the envelope[7] of the extremals $z = \varphi(x, \alpha)$, $x_1 \le x \le x_2$, emanating from the point $P_1 = (x_1, z_1)$, and the conjugate point $P_1^* = P^*(\alpha_0)$ lies both on $\mathscr{E}$ and on the extremal described by $u(x)$. More general, the condition $\varphi_\alpha(x, \alpha) = 0$ determines points on the extremal $\mathscr{C}(\alpha) = \{(x, z) \colon z = \varphi(x, \alpha), x \in I\}$ which are conjugate to P_1.

If $\varphi_{\alpha\alpha}(x_1^*, \alpha_0) = 0$, the solution set $\mathscr{E}$ of (1) can be rather degenerate, for instance, a curve with a cusp at P_1^*, or even an isolated point (nodal point). In all cases, whether degenerate or not, we shall denote $\mathscr{E}$ as envelope of the family of curves $\mathscr{C}(\alpha)$. Then we can state:

Theorem 1. *Suppose that $\mathscr{C}(\alpha) = \{(x, z) \colon z = \varphi(x, \alpha), x \in I\}$, $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta)$, is a family of extremals of $\mathscr{F}$ emanating from a fixed point $P_1(x_1, z_1)$, i.e., $\varphi(x_1, \alpha) = z_1$. Assume also that $\varphi(x, \alpha)$ is of class C^3 and that $\varphi_\alpha'(x_1, \alpha) = \varphi_{\alpha x}(x_1, \alpha) \ne 0$ for $|\alpha - \alpha_0| < \delta$. Then the envelope $\mathscr{E}$ of the family $\{\mathscr{C}(\alpha)\}$ consists of all points $P^*(\alpha) \in \mathscr{C}(\alpha)$ conjugate to P_1, α varying in $(\alpha_0 - \delta, \alpha_0 + \delta)$. To be precise, if $P^*(\alpha) \in \mathscr{C}(\alpha) \cap \mathscr{E}$, and if $\mathscr{E}$ is regular at $P^*(\alpha)$, then $P^*(\alpha)$ is a conjugate point to P_1 with respect to the extremal $\mathscr{C}(\alpha)$, and $\mathscr{E}$ is tangent to $\mathscr{C}(\alpha)$ at $P^*(\alpha)$.*

If we imagine the extremals $\mathscr{C}(\alpha)$ as a pencil of light rays, we can interpret the envelope $\mathscr{E}$ as its *caustics*.

[7] We recall that the envelope $\mathscr{E}$ of the family $\varphi(x, \alpha)$ is defined by $\mathscr{E} := \{(x, z) \colon$ there is an α such that $z = \varphi(x, \alpha)$ and $\varphi_\alpha(x, \alpha) = 0\}$.