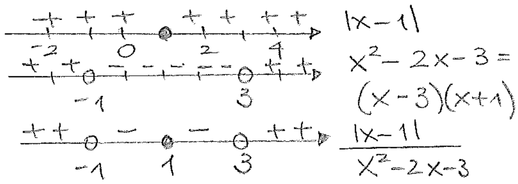
Università degli Studi di Trento – Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
ESAME SCRITTO DI ANALISI MATEMATICA
a.a. 2016 – 2017 – Rovereto, 13 gennaio 2017

FILA (A)

1) i) $A = \left\{ x \in \mathbb{R} : \dfrac{|x-1|}{x^2-2x-3} < 0 \right\}$

$= \,]-1,3[\setminus \{1\}$.



$B = \left\{ x \in \mathbb{R} : e^{x^2-3x} \geq \dfrac{1}{e^{4x}} \right\} = \,]-\infty,-1] \cup [0,+\infty[$.

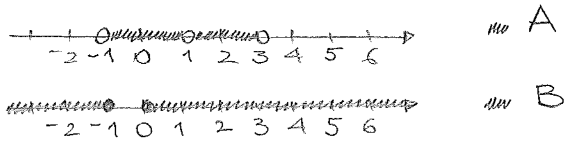
Infatti $e^{x^2-3x} \geq \dfrac{1}{e^{4x}} \iff e^{x^2-3x} \geq e^{-4x} \underset{e^x \uparrow}{\iff} x^2-3x \geq -4x$

$\iff x^2+x \geq 0$

$\iff x(x+1) \geq 0$

$x \in \,]-\infty,-1] \cup [0,+\infty[$.

Abbiamo

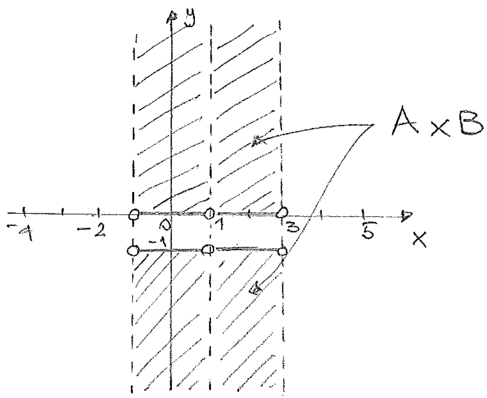


Né A, né B è un intervallo.

ii) $A \cup B = \mathbb{R}$; $A \cap B = [0,3[\setminus \{1\}$.

iii) A è limitato inferiormente (infatti, $\forall x \in A$ si ha $x \geq -1$)
è limitato superiormente (infatti, $\forall x \in A$ si ha $x \leq 3$),
e quindi è limitato.

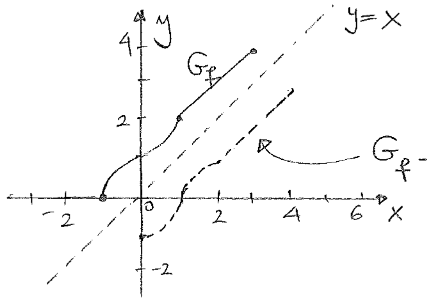
B non è limitato (non è nemmeno limit. inferior. o limitato superior.)

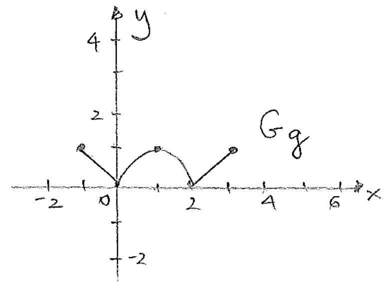


2) $f, g : [-1,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} \sqrt[3]{x+1} & \text{se } -1 \le x < 0 \\ 2^x & \text{se } 0 \le x \le 1 \\ x+1 & \text{se } 1 < x \le 3 \end{cases}, \qquad g(x) = \begin{cases} |x-1|-1 & \text{se } x \in [-1,3] \setminus [0,2] \\ -(x-1)^2+1 & \text{se } x \in [0,2] \end{cases}$$

i)





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ii) f è strett. crescente su $[-1,3]$, quindi iniettiva.

$\underline{f([-1,3]) = [0,4]}$. $f^{-1} : [0,4] \to [-1,3]$ rappresentato sopra.

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iii) $\min\limits_{[-1,3]} g = \underline{\underline{0}}$ $\underline{x \in \{0, 2\} \text{ pt. di minimo}}$.

$\max\limits_{[-1,3]} g = \underline{\underline{1}}$ $\underline{x \in \{-1, 1, 3\} \text{ pt. di massimo}}$.

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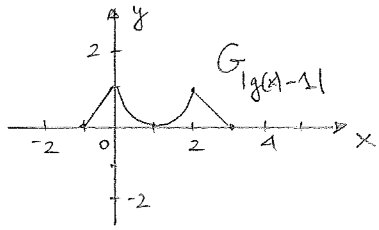
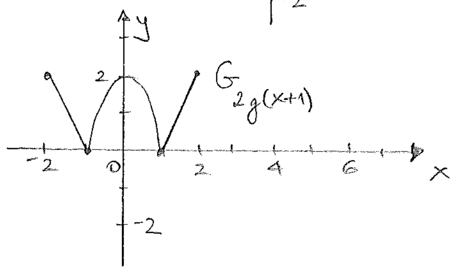
iv) $x \longmapsto |g(x)-1|$, $x \in [-1,3]$



$x \longmapsto 2g(x+1)$, $x \in [-2,2]$



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v) $(f \circ g)(x) =$

$\forall x \in [-1,3] \Rightarrow g(x) \in [0,1] \subset \text{dom} f$ (ricordiamo che $f(x) = 2^x$ se $x \in [0,1]$)

$$= \begin{cases} 2^{|x-1|-1} & \text{se } x \in [-1,3] \setminus [0,2] \\ 2^{-(x-1)^2+1} & \text{se } x \in [0,2] \end{cases}$$

■

3) i) $\lim\limits_{x \to -\infty} \dfrac{\overset{\to 0}{e^x} + x^2}{\log|x| + 3x^2} = \underline{\underline{\dfrac{1}{3}}}$ ($\log|x|$ trascurabile rispetto a $3x^2$)

$\lim\limits_{x \to 0} \dfrac{\log(1+2x)}{3x} = \lim\limits_{x \to 0} \underbrace{\dfrac{\log(1+2x)}{2x}}_{\to 1} \cdot \dfrac{2}{3} = \underline{\underline{\dfrac{2}{3}}}$

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ii) $\displaystyle\sum_{n=1}^{4} \int_{n}^{n+1} (e^x+1)\,dx = \int_1^5 (e^x+1)\,dx = \left[e^x + x\right]_1^5 = (e^5+5)-(e+1) =$

$= \underline{\underline{e^5 - e + 4}}.$ □

iii) $f(x) = -x^2+4x+1$ L'eq. retta tg. al grafico di f in $(0,1)$ è
$(x_0 = 0,\ f(x_0) = 1)$

$$y = f(x_0) + f'(x_0)(x-x_0).$$

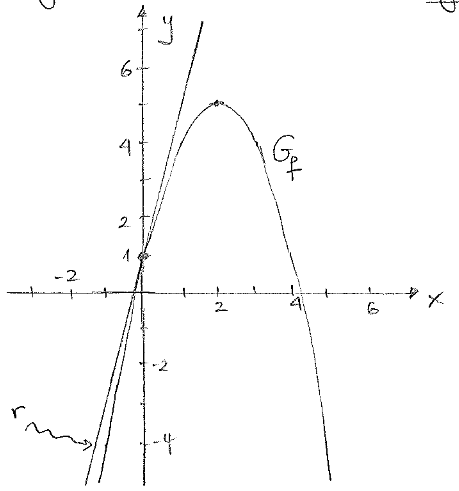
Poichè $f'(x) = -2x+4$

$f'(0) = 4$ si ha $y = 1 + 4(x-0) \Rightarrow \underline{\underline{y = 4x+1}}.$

$f(x) = -(x^2-4x)+1$

$= -(x-2)^2 + 4 + 1$

$= -(x-2)^2 + 5$



4) $F(x) = \displaystyle\int_0^x \underbrace{\frac{t}{t^2+1}}_{f(t)}\,dt$ su $[0,1]$.

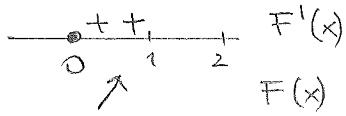
i) Dal teorema fondamentale del calcolo integrale si ha $F'(x) = f(x)$ su $[0,1]$,

ossia $F'(x) = \dfrac{x}{x^2+1} \geq 0$ su $[0,1]$, e $F'(x) > 0$ su $]0,1]$.

Quindi F è strettamente crescente su $[0,1]$. □

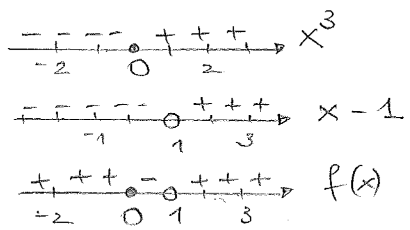
ii) $x=0$ pt. di minimo di F su $[0,1]$,

$x=1$ pt. di massimo di F su $[0,1]$.



5) $\boxed{f(x) = \dfrac{x^3}{x-1}}$

- dom $f = \mathbb{R}\setminus\{1\} =]-\infty,1[\,\cup\,]1,+\infty[$.
- segno



- $\displaystyle\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^{\cancel{3}\,2}}{\cancel{x}(1-\frac{1}{x})} = \underline{\underline{+\infty}}$

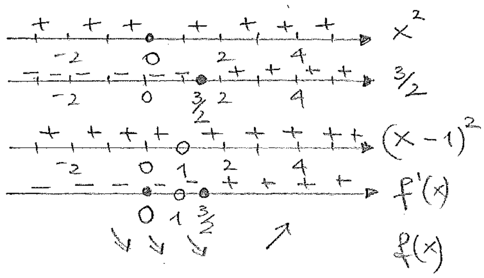
$\displaystyle\lim_{x\to1^-} f(x) = \underline{\underline{-\infty}}$ $\displaystyle\lim_{x\to1^+} f(x) = \underline{\underline{+\infty}}$ $x=1$ asintoto verticale per f (da destra e da sinistra)

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^{\cancel{3}\,2}}{\cancel{x}(1-\frac{1}{x})} = \underline{\underline{+\infty}}\,.$$

- dom f' = dom f $\qquad f'(x) = \dfrac{3x^2(x-1) - x^3}{(x-1)^2} = \dfrac{x^2(3x-3-x)}{(x-1)^2}$

$$= \frac{x^2(2x-3)}{(x-1)^2}$$

Allora $f'(x) = 0 \iff x = 0,\ x = 3/2$



$x = 0$ è un pt. con tangente orizzontale, ma non un pt. di massimo loc., e nemmeno un pt. di minimo loc.

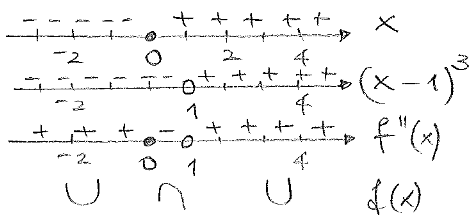
$x = 3/2$ pt. di min. loc. stretto per f $\qquad f\left(\frac{3}{2}\right) = \dfrac{\frac{27}{8}}{\frac{1}{2}} = \dfrac{27}{\cancel{8}\,4}\cdot\cancel{2} = \underline{\underline{\dfrac{27}{4}}}$

- dom f'' = dom f

$$f''(x) = \left(\frac{2x^3-3x^2}{(x-1)^2}\right)' = \frac{(6x^2-6x)(x-1)^{\cancel{2}} - (2x^3-3x^2)2\cancel{(x-1)}}{(x-1)^{\cancel{4}\,3}} =$$

$$= \frac{6x^3 - \cancel{6x^2} - 6x^2 + 6x - 4x^3 + \cancel{6x^2}}{(x-1)^3} = \frac{2x^3-6x^2+6x}{(x-1)^3}$$

$$= \frac{2x\,(x^2-3x+3)}{(x-1)^3}$$



$x = 0$ pt. di flesso per f

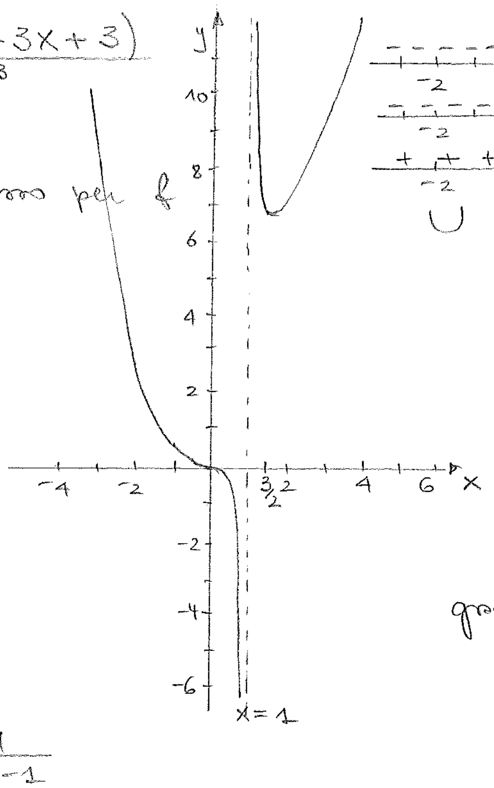


grafico qualitativo di f

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ii) $f(x) \overset{?}{=} x^2 + x + 1 + \dfrac{1}{x-1}$

$$\overset{?}{=} \frac{x^3 - \cancel{x^2} + \cancel{x^2} - \cancel{x} + \cancel{x} - \cancel{1} + \cancel{1}}{x-1} = \frac{x^3}{x-1} \qquad \text{OK.}$$

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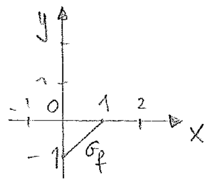
iii) $\int_2^3 f(x)\,dx \underset{ii)}{=} \int_2^3 \left(x^2+x+1+\frac{1}{x-1}\right)dx = \left[\frac{x^3}{3}+\frac{x^2}{2}+x+\log|x-1|\right]_2^3$

$= \left(9+\frac{9}{2}+3+\log 2\right) - \left(\frac{8}{3}+2+2\right) = 8+\frac{9}{2}-\frac{8}{3}+\log 2$

$= \frac{48-27-16}{6}+\log 2 = \underline{\underline{\frac{5}{6}+\log 2}}.$

6) i) Se $f'(x)>0$ su $[0,1]$, allora $f(x)>0$ su $[0,1]$: F

Es. $f(x) = x-1$ su $[0,1]$:



allora $f'(x)>0$ su $[0,1]$, ma $f(x)\le 0$ su $[0,1]$.

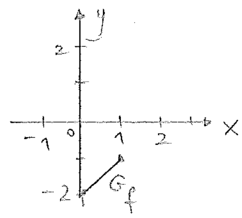
ii) Se $f(0)=2$ e $f'(x)>0$ su $[0,1]$, allora $f(x)>0$ su $[0,1]$: V.

Infatti $f'(x)>0$ su $[0,1] \Rightarrow f$ crescente su $[0,1]$.

Poiché $f(0)=2$, risulta quindi $f(x)>2>0$ su $[0,1]$.

iii) Se $f(x)<0$ su $[0,1]$, allora $f'(x)<0$ su $[0,1]$: F

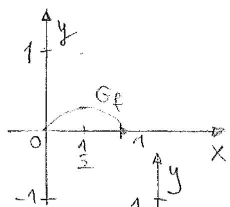
Es. $f(x) = x-2$ su $[0,1]$:



allora $f(x)<0$ su $[0,1]$, ma $f'(x)=1>0$ su $[0,1]$.

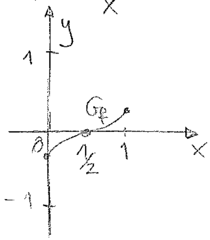
iv) Se $f'\left(\frac{1}{2}\right)=0$, allora $x_0=\frac{1}{2}$ è un pt. di minimo : F

Es.



$f(x) = -\left(x-\frac{1}{2}\right)^2+\frac{1}{4}$ $\qquad x_0=\frac{1}{2}$ pt. di massimo.

$f'\left(\frac{1}{2}\right)=0$;



Oppure $f(x) = \left(x-\frac{1}{2}\right)^3$, $f'\left(\frac{1}{2}\right)=0$ $\qquad x_0=\frac{1}{2}$ non è né pt. di minimo, né pt. di massimo.

<u>NOTA</u>: una giustificazione grafica delle risposte date sopra è sufficiente!

FILA (B)

1) i) $A=\{x\in\mathbb{R}: 2^{x^2+3x}\le \frac{1}{4^x}\} = \underline{\underline{[-5,0]}}$.

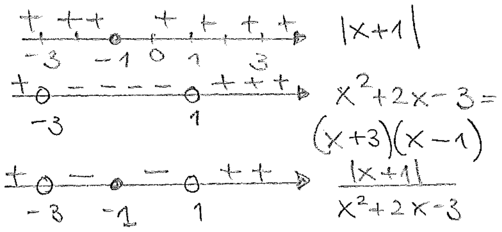
Infatti $2^{x^2+3x}\le 4^{-x} \iff 2^{x^2+3x}\le 2^{-2x} \underset{2^x\uparrow}{\iff} x^2+3x\le -2x$

$x^2+5x\le 0$

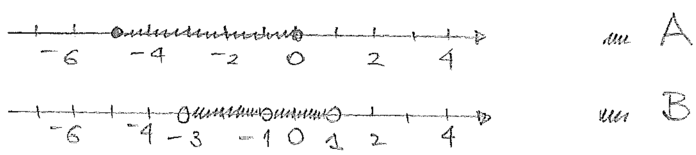
$\Rightarrow x\in[-5,0]$

$B=\left\{x\in\mathbb{R}: \frac{|x+1|}{x^2+2x-3}<0\right\} =$

$= \underline{\underline{]-3,1[\setminus\{-1\}}}$



Abbiamo

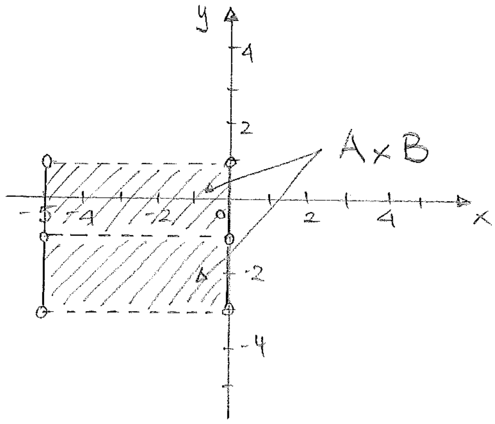


A è un intervallo, B non lo è.

ii) $\underline{\underline{A\cup B=[-5,1[}}$; $\underline{\underline{A\cap B=]-3,0]\setminus\{-1\}}}$.

iii) A e B sono insiemi limitati (infatti, $\forall x\in A$, $-5\le x\le 0$

$\forall x\in B$, $-3\le x\le 1$)

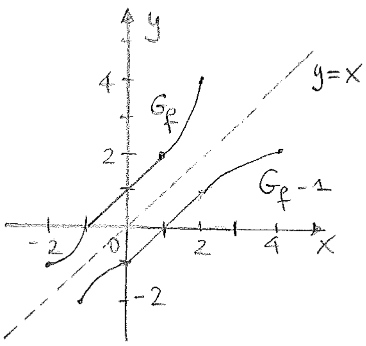


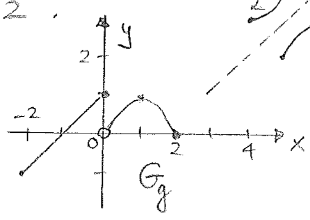
2) $f,g: [-2,2]\to\mathbb{R}$,

$$f(x)=\begin{cases}\sqrt[3]{x+1} & \text{se } -2\le x<-1\\ x+1 & \text{se } -1\le x\le 1\\ 2^x & \text{se } 1<x\le 2.\end{cases}$$

$$g(x)=\begin{cases}2-|x-1| & \text{se } x\in[-2,0]\\ -(x-1)^2+1 & \text{se } x\in]0,2]\end{cases}$$

i)





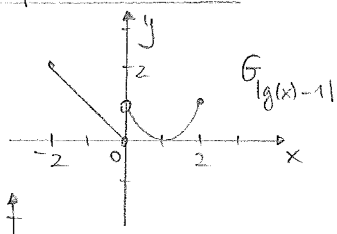
ii) f è strettamente crescente su $[-2,2]$, quindi iniettiva.

$\underline{\underline{f([-2,2]) = [-1,4]}}$. $\quad f^{-1} : [-1,4] \to [-2,2]$ rappresentata sopra

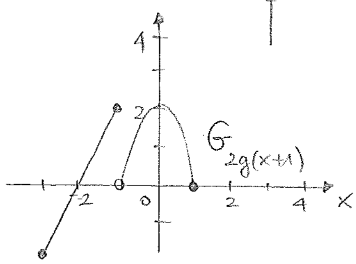
iii) $\min\limits_{[-2,2]} g = \underline{\underline{-1}}$ $\quad$ $\underline{x=-2 \text{ pt. di minimo.}}$

$\max\limits_{[-2,2]} g = \underline{\underline{1}}$ $\quad$ $\underline{x \in \{0, 1\} \text{ pt. di massimo.}}$

iv) $\underset{[-2,2]}{\underset{\Uparrow}{x}} \longmapsto |g(x)-1|$



$\underset{[-3,1]}{\underset{\Uparrow}{x}} \longmapsto 2g(x+1)$



v) $(f \circ g)(x) =$

$x \in [-2,2] \Rightarrow g(x) \in [-1,1] \subset \operatorname{dom} f$

$$= \begin{cases} (2-|x-1|)+1 & \text{se } x \in [-2,0] \\ (-(x-1)^2+1)+1 & \text{se } x \in]0,2] \end{cases}$$

ricordiamo che
$f(x) = x+1$ se $x \in [-1,1]$

$$= \begin{cases} 3-|x-1| & \text{se } x \in [-2,0] \\ -(x-1)^2+2 & \text{se } x \in]0,2]. \end{cases}$$

3) i) $\lim\limits_{x\to-\infty} \dfrac{\overset{\to 0}{e^x} + 3x^3}{\underbrace{\log|x|}_{\text{trascurabile rispetto a } 2x^3} + 2x^3} = \underline{\underline{\dfrac{3}{2}}}$

$\lim\limits_{x\to 0} \dfrac{e^{2x}-1}{4x} = \lim\limits_{x\to 0} \underbrace{\dfrac{e^{2x}-1}{2x}}_{\to 1} \cdot \dfrac{1}{2} = \underline{\underline{\dfrac{1}{2}}}$

ii) $\displaystyle\sum_{n=1}^{5} \int_n^{n+1} (x-e^x)\,dx = \int_1^6 (x-e^x)\,dx = \left[\frac{x^2}{2} - e^x\right]_1^6 = \left(\frac{36}{2} - e^6\right) - \left(\frac{1}{2} - e\right)$

$= \underline{\underline{-e^6 + e + \dfrac{35}{2}}}.$

iii) $f(x) = -x^2-4x+1$ L'eq. retta tg al grafico di f in $(\underset{x_0}{0}, \underset{f(x_0)}{1})$ è $y = f(x_0) + f'(x_0)(x-x_0)$

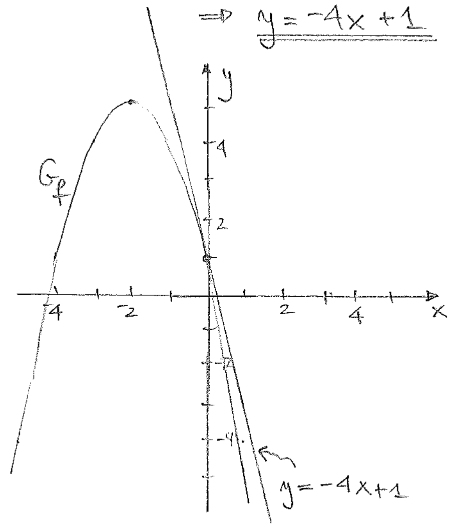
Poichè $f'(x) = -2x-4$ $\quad f'(0) = -4$ e si ha $\quad y = 1 - 4(x-0)$

$\Rightarrow \underline{y = -4x+1}$.

$f(x) = -(x^2+4x)+1$

$= -(x+2)^2 + 4 + 1$

$= -(x+2)^2 + 5$



4) $F(x) = \int_0^x \frac{-t}{t^2+1}\,dt$ su $[0,1]$.

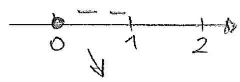
$f(t)$

i) Dal teorema fondamentale del calcolo integrale si ha $F'(x) = f(x)$ su $[0,1]$, ovvia $F'(x) = \frac{-x}{x^2+1} \leq 0$ su $[0,1]$, e $F'(x) < 0$ su $]0,1]$.

Quindi F è strett. decrescente su $[0,1]$. □

ii) $x = 0$ pt. di massimo di F su $[0,1]$

$x = 1$ pt. di minimo di F su $[0,1]$



5) i) $\boxed{f(x) = \frac{x^3}{1-x}}$ questa funzione è uguale a $-$ la funzione f studiata nell'esercizio 5) Fila A.

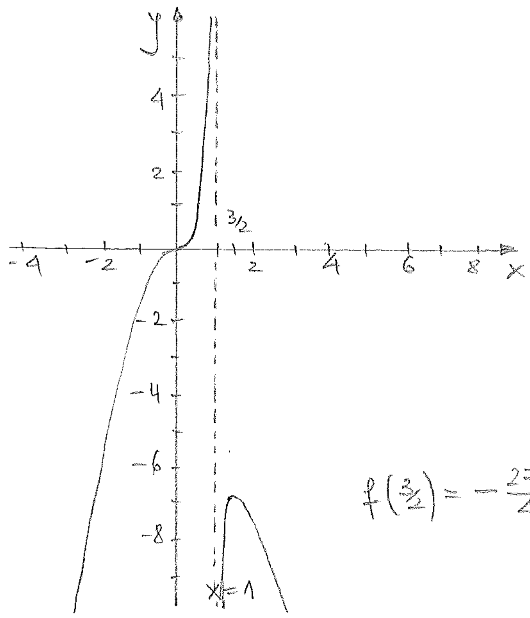


grafico qualitativo di f

$f\left(\frac{3}{2}\right) = -\frac{27}{4}$

□

ii) $f(x) \overset{?}{=} -x^2-x-1+\frac{1}{1-x}$

$\overset{?}{=} \frac{\cancel{-x^2}+x^3\cancel{-x}+\cancel{x^2}\cancel{-1}+\cancel{x}+\cancel{1}}{1-x} = \frac{x^3}{1-x}$ OK.

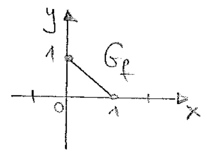
iii) $\int_2^3 f(x)\,dx = \int_2^3 \left(-x^2-x-1+\frac{1}{1-x}\right)dx = \left[-\frac{x^3}{3}-\frac{x^2}{2}-x-\log|1-x|\right]_2^3$

$= \left(-9-\frac{9}{2}-3-\log 2\right)-\left(-\frac{8}{3}-2-2\right) = -8-\frac{9}{2}+\frac{8}{3}-\log 2$

$= \underline{-\frac{5}{6}-\log 2}$

6) i) Se $f'(x)<0$ su $[0,1]$, allora $f(x)<0$ su $[0,1]$: F

Es. $f(x)=-x+1$ su $[0,1]$; allora $f'(x)<0$ su $[0,1]$, ma $f(x)\geq 0$ su $[0,1]$

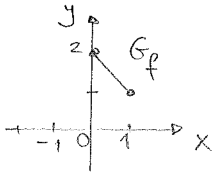


ii) Se $f(0)=2$ e $f'(x)>0$ su $[0,1]$, allora $f(x)>0$ su $[0,1]$: V

(vedi Es.6 ii) Fila A).

iii) Se $f(x)>0$ su $[0,1]$, allora $f'(x)>0$ su $[0,1]$: F

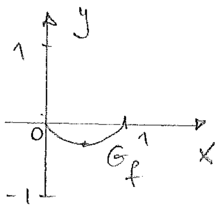
Es. $f(x)=-x+2$ su $[0,1]$



allora $f(x)>0$ su $[0,1]$, ma $f'(x)<0$ su $[0,1]$.

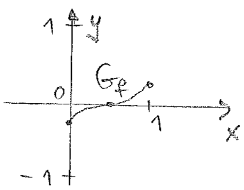
iv) Se $f'(\frac{1}{2})=0$, allora $x_0=\frac{1}{2}$ è un pt. di massimo: F

Es.



$f(x)=(x-\frac{1}{2})^2-\frac{1}{4}$ $\quad x_0=\frac{1}{2}$ pt. di minimo.

$f'(\frac{1}{2})=0$



oppure $f(x)=(x-\frac{1}{2})^3$ $\quad x_0=\frac{1}{2}$ non è né pt. di minimo, né pt. di massimo!

$f'(\frac{1}{2})=0$

NOTA: una giustificazione grafica delle risposte date sopra è sufficiente!