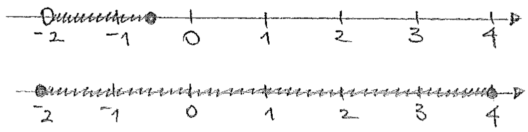
Università di Trento – Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
ESAME SCRITTO DI ANALISI MATEMATICA
a.a. 2015 - 2016 – Rovereto, 1 febbraio 2016

FILA (A)

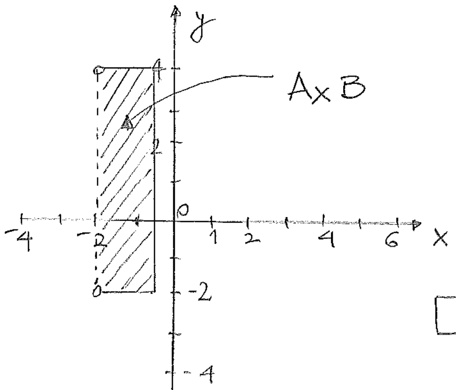
1) i) $A = \{x \in \mathbb{R} : (\log_2 4)\log(x+2) - \log(x+5) \leq \log(-x)\} = \left]-2, -\frac{1}{2}\right]$.

$B = \{x \in \mathbb{R} : 3|x+1| - x^2 \geq -1\} = [-2, 4]$

(vedi Seconda Prova Intermedia, 11/01/2016 Fila (A) Es. 1, Fila (B) Es. 1)



ii) $A \cup B = B$, $\mathbb{R} \setminus A = \left]-\infty, -2\right] \cup \left]-\frac{1}{2}, +\infty\right[$.



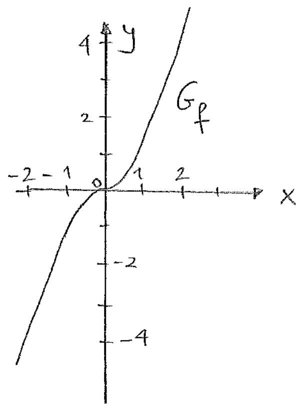
iii) A e B sono insiemi limitati.

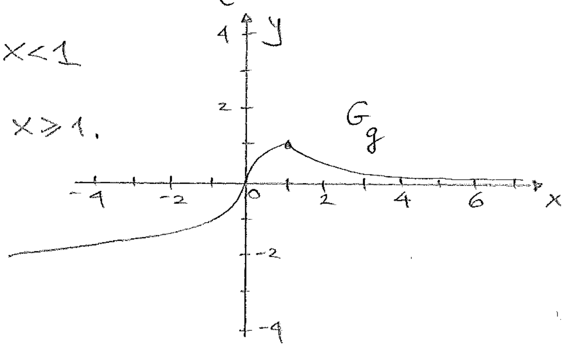
min B = -2

max B = 4.

2) i) $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = x|x| = \begin{cases} x^2 & se\ x \geq 0 \\ -x^2 & se\ x < 0 \end{cases}$

$g(x) = \begin{cases} \sqrt[3]{x} & se\ x < 1 \\ 1/x^3 & se\ x \geq 1. \end{cases}$





ii) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \sqrt[3]{x} = 1$

$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \frac{1}{x^3} = 1 = g(1) \Rightarrow$ g è continua in $x = 1$.

iii) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h^2 - 0}{h} = \lim_{h \to 0^-} (-h) = 0$.

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0.$$

Quindi, essendo $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ ed è finito, si ha che f è derivabile in $x=0$. □

iv) f è iniettiva poiché strettamente crescente su $\mathbb{R}$ ($f'(x)>0 \ \forall x\neq 0$ e $=0 \iff x=0$).

$\operatorname{im} g = g(\mathbb{R}) =]-\infty, 1]$. □

v) $\min\limits_{[-1,2]} f = -1$ $\quad x=-1$ pt. di minimo.

$\max\limits_{[-1,2]} f = 4$ $\quad x=2$ pt. di massimo. □

vi) $$(g\circ f)(x) \underset{x\in\mathbb{R}}{=} g(f(x)) = \begin{cases} \sqrt[3]{f(x)} & \text{se } f(x)<1 \\ \frac{1}{f(x)^3} & \text{se } f(x)\geq 1 \end{cases}$$

$$= \begin{cases} \sqrt[3]{x|x|} & \text{se } x<1 \\ \frac{1}{(x|x|)^3} & \text{se } x\geq 1. \end{cases}$$ ■

3) i) $$\int_1^2 \left(e^{3x}+\frac{1}{x^2}\right)dx = \frac{1}{3}\int_1^2 3e^{3x}\,dx + \int_1^2 \frac{1}{x^2}dx = \frac{1}{3}\left[e^{3x}\right]_1^2 + \left[-\frac{1}{x}\right]_1^2$$

$$= \frac{1}{3}\left(e^6-e^3\right)+\frac{1}{2}.$$

$$\lim_{x\to 0} \frac{\log(1+3x)}{2x} = \lim_{x\to 0} \underbrace{\frac{\log(1+3x)}{3x}}_{\to 1}\cdot\frac{3}{2} = \frac{3}{2}.$$

$$\lim_{x\to+\infty} \frac{2^{-x}+x^3}{4x^3+\log x} = \frac{1}{4}.$$ □

ii) $$\log\left(\frac{5}{4}\right)-\log\left(\frac{10}{9}\right)+\dots-\log\left(\frac{122}{121}\right) = \sum_{n=2}^{11}(-1)^n\log\left(\frac{n^2+1}{n^2}\right).$$ ■

4) $f(x) = x\log(1+x) \quad x\in]-1,+\infty[$.

i) $F(x) = \left(\frac{x^2}{2}-\frac{1}{2}\right)\log(1+x)-\frac{x^2}{4}+\frac{x}{2}$ è derivabile e

$$F'(x) = x\log(1+x)+\left(\frac{x^2}{2}-\frac{1}{2}\right)\cdot\frac{1}{1+x}-\frac{x}{2}+\frac{1}{2} = x\log(1+x)+\frac{1}{2}\frac{(x-1)\cancel{(x+1)}}{\cancel{1+x}}$$
$$-\frac{1}{2}(x-1) = x\log(1+x) = f(x) \quad \forall x\in]-1,+\infty[.$$ □

ii) L'insieme di tutte le primitive di f $= \int f(x)\,dx = \{F(x)+c \,;\, c\in\mathbb{R}\}$.

Dobbiamo infine determinare $c\in\mathbb{R}$, tale che $G(x)=F(x)+c$ soddisfi $G(0)=4$. Ora $4=G(0)=F(0)+c = 0+c \iff c=4$.

Quindi $G(x)=F(x)+4$ è la primitiva ricercata. □

iii) $f(x)=x\log(1+x)$; ora $f(1)=\log 2$; inoltre $f'(x)=\log(1+x)+\frac{x}{1+x}$ e quindi $f'(1)=\log 2+\frac{1}{2}$.

L'eq. della retta tg r è quindi $y=\log 2+\left(\log 2+\frac{1}{2}\right)(x-1)$. □

iv) $\int_0^1 f(x)\,dx = \left[F(x)\right]_0^1 = F(1)-F(0) = \frac{1}{4}$. ■

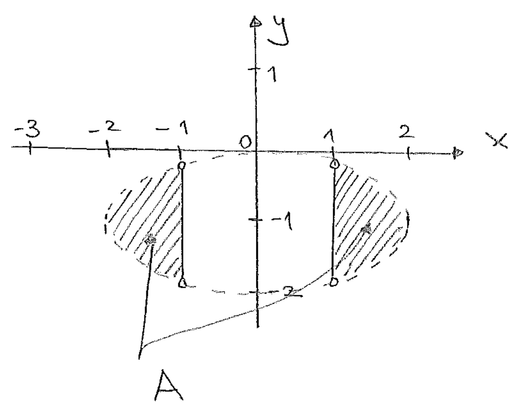
5) Studio di $\boxed{f(x)=\dfrac{x}{\log x - 1}}$ (Vedi Seconda Prova Intermedia, 11/01/2016 FILA A, Es. 5) ■

↳ e rappr. grafica di $f(|x|)$.

6) $\begin{cases} x^2+4y^2+8y<0 \\ |x|\geq 1 \end{cases}$

$$x^2+4(y^2+2y)<0 \iff x^2+4(y+1)^2-4<0$$
$$\iff x^2+4(y+1)^2<4$$
$$\iff \frac{x^2}{4}+(y+1)^2<1$$

$|x|\geq 1 \iff x\leq -1$ o $x\geq 1$ (y libero in $\mathbb{R}$)



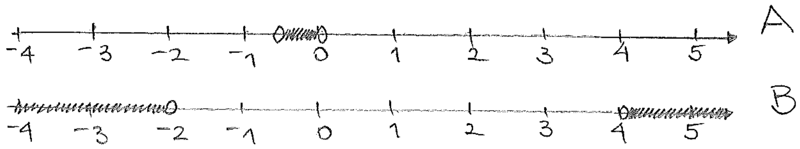
■

FILA (B)

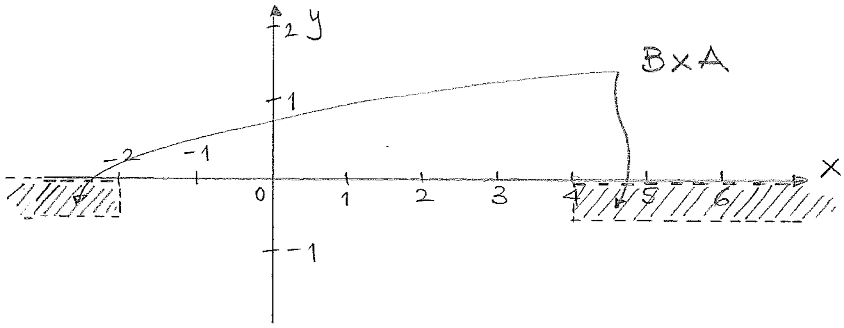
1) i) $A = \{x \in \mathbb{R} : (\log_3 9) \log(x+2) - \log(x+5) > \log(-x)\} = \left]-\frac{1}{2}, 0\right[$.

$B = \{x \in \mathbb{R} : 3|x+1| - x^2 < -1\} = \left]-\infty, -2\right[\cup \left]4, +\infty\right[$.

(vedi Seconda Prova Intermedia, 11/01/2016, Fila (B) Es. 1, Fila (A) Es. 1).



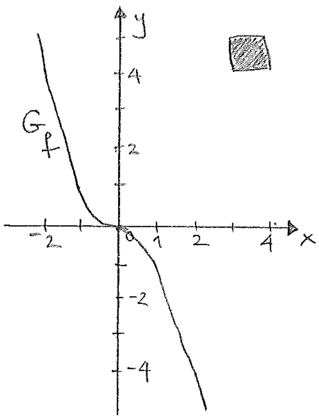
ii) $A \cup B = \left]-\infty, -2\right[\cup \left]-\frac{1}{2}, 0\right[\cup \left]4, +\infty\right[$ $\qquad$ $\mathbb{R} \setminus B = [-2, 4]$.



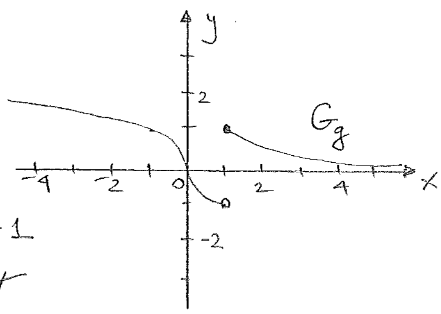
iii) A è limitato, B non lo è.

$\nexists \min A$, $\nexists \max A$.

2) $f, g : \mathbb{R} \to \mathbb{R}$ $\qquad$ $f(x) = -x|x| = \begin{cases} -x^2 & \text{se } x \geq 0 \\ x^2 & \text{se } x < 0 \end{cases}$



$g(x) = \begin{cases} -\sqrt[3]{x} & \text{se } x < 1 \\ \frac{1}{x} & \text{se } x \geq 1. \end{cases}$



ii) $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \left(-\sqrt[3]{x}\right) = -1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \neq$

$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} \frac{1}{x} = 1 = g(1)$. $\qquad$ g non è continua in $x = 1$. □

iii) $\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^2 - 0}{h} = \lim_{h \to 0^-} h = 0$.

$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{-h^2 - 0}{h} = \lim_{h \to 0^+} (-h) = 0$.

Quindi, essendo $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$ ed è finito, si ha che f è derivabile in $x=0$. □

iv) f è iniettiva poiché strett. decrescente su $\mathbb{R}$ ($f'(x)<0 \ \forall x\neq 0$ e $=0 \iff x=0$).

$\operatorname{Im} g = g(\mathbb{R}) = \underline{]-1,+\infty[}$. □

v) $\min_{[-3,1]} f = \underline{-1}$ $\quad$ $\underline{x = 1 \text{ pt. di minimo.}}$

$\max_{[-3,1]} f = \underline{9}$ $\quad$ $\underline{x = -3 \text{ pt. di massimo.}}$ □

vi) $(g\circ f)(x) \underset{x\in\mathbb{R}}{=} g(f(x)) = \begin{cases} -\sqrt[3]{f(x)} & \text{se } f(x)<1 \\ \frac{1}{f(x)} & \text{se } f(x)\geq 1 \end{cases}$

$= \begin{cases} -\sqrt[3]{-x|x|} & \text{se } x>-1 \\ \frac{1}{-x|x|} & \text{se } x\leq -1. \end{cases}$ ■

3) i) $\int_1^2 \left(e^{2x} - \frac{1}{x^2}\right)dx = \frac{1}{2}\int_1^2 2e^{2x}\,dx - \int_1^2 \frac{1}{x^2}dx = \frac{1}{2}\left[e^{2x}\right]_1^2 + \left[\frac{1}{x}\right]_1^2 =$

$= \underline{\frac{1}{2}\left(e^4 - e^2\right) - \frac{1}{2}}$.

$\lim_{x\to 0} \frac{e^{2x}-1}{3x} = \lim_{x\to 0} \underbrace{\frac{e^{2x}-1}{2x}}_{\to 1} \cdot \frac{2}{3} = \underline{\frac{2}{3}}$.

$\lim_{x\to+\infty} \frac{e^{-x}-x^2}{5x^2+\log x} = \underline{-\frac{1}{5}}$. □

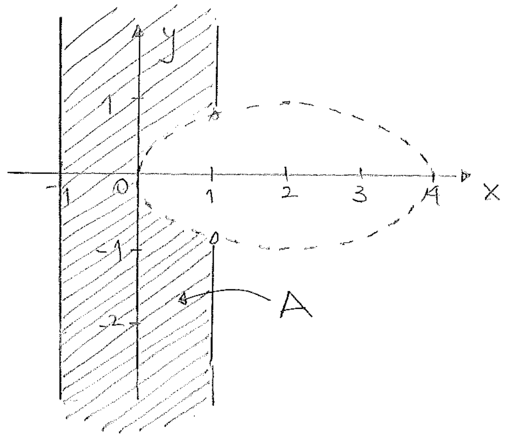
ii) $-e^{-\frac{5}{4}} + e^{-\frac{10}{9}} - \dots - e^{-\frac{145}{144}} = \underline{\sum_{n=2}^{12} (-1)^{n+1} e^{-\frac{n^2+1}{n^2}}}$ ■

4) $\begin{cases} x^2-4x+4y^2>0 \\ |x|\leq 1 \end{cases}$ $\quad$ $(x-2)^2-4+4y^2>0$

$\iff (x-2)^2+4y^2>4$

$\iff \frac{(x-2)^2}{4}+y^2>1$

Inoltre $|x|\leq 1 \iff -1\leq x\leq 1$ (mentre $y\in\mathbb{R}$ libera)



5) Studio di $\boxed{f(x) = \frac{x}{1-\log x}}$ $\hookrightarrow$ e rappr. grafica di $f(|x|)$

Vedi Seconda Prova Intermedia, 11/01/2016 FILA (B), Es. 5

6) i) Vedi FILA (A), Es. 4 i)

ii) 〃 ii).

$\underline{G(x) = F(x) + 2}$ è la primitiva ricercata.

iii) $f(x) = x\log(1+x)$; ora $f(2) = 2\log 3$; inoltre

$$f'(x) = \log(1+x) + \frac{x}{1+x}$$

e quindi $f'(2) = \log 3 + \frac{2}{3}$.

L'eq. della retta tg r è quindi $\underline{y = 2\log 3 + \left(\log 3 + \frac{2}{3}\right)(x-2)}$.

iv) $\int_0^1 f(x)\,dx = \left[F(x)\right]_0^1 = F(1) - F(0) = \underline{\frac{1}{4}}$.