Università di Trento – Dip. di Psicologia e Scienze Cognitive
CdL in Scienze e Tecniche di Psicologia Cognitiva
ESAME SCRITTO DI ANALISI MATEMATICA
a.a. 2015-2016 – Rovereto, 7 giugno 2016

FILA (A)

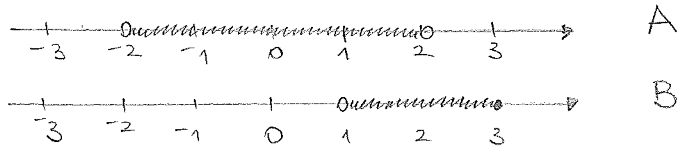
1) i) $A = \{x \in \mathbb{R} : \frac{|x^2-1|-3}{|x|+2} < 0\} = \{x \in \mathbb{R} : |x^2-1|-3 < 0\}$

Ora $|x^2-1| < 3 \iff -3 < x^2-1 < 3 \iff \underset{\uparrow \text{ sempre}}{-2 < x^2} < 4 \iff -2 < x < 2$.

Quindi $\underline{\underline{A =]-2,2[}}$.

$B = \{x \in \mathbb{R} : \log_2(x-1) \leq 1\}$: $\begin{cases} x-1 > 0 \\ x-1 \leq 2 \end{cases}$ (altrimenti non è def. $\log_2(x-1)$) (poiché $\log_2 2 = 1$, e usando il fatto che $\log_2 x \uparrow$)

Quindi $\underline{\underline{B =]1,3]}}$.

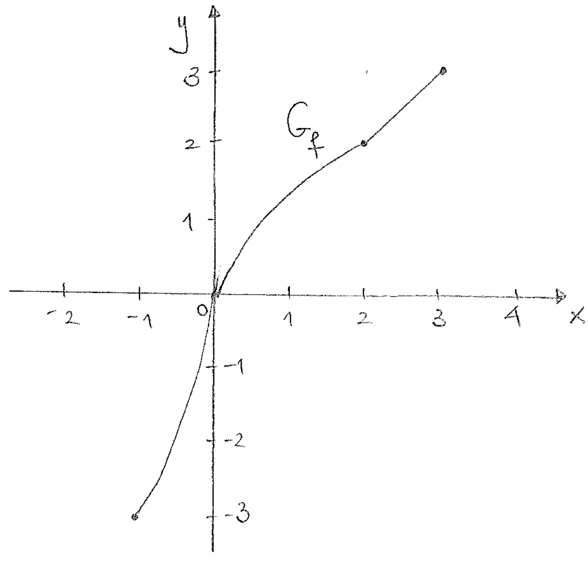


ii) $A \cup B = \underline{\underline{]-2,3]}}$; $A \cap B = \underline{\underline{]1,2[}}$; $A \setminus B = \underline{\underline{]-2,1]}}$.

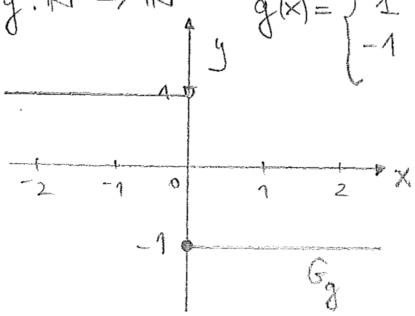
iii) A e B sono insiemi $\underline{\underline{\text{limitati}}}$; $\underline{\underline{\min B \nexists}}$, $\underline{\underline{\max B = 3}}$.

2) i) $f : [-1,3] \to \mathbb{R}$

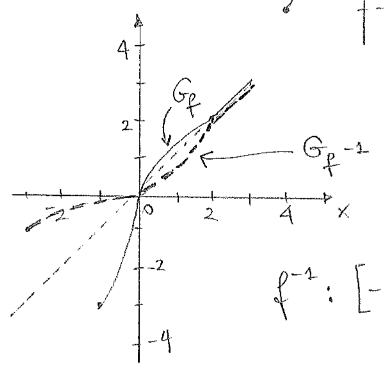
$$f(x) = \begin{cases} 3\sqrt[3]{x} & \text{se } -1 \leq x \leq 0 \\ 2\log_3(x+1) & \text{se } 0 < x < 2 \\ x & \text{se } 2 \leq x \leq 3 \end{cases}$$



$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = \begin{cases} 1 & \text{se } x < 0 \\ -1 & \text{se } x \geq 0 \end{cases}$



ii) $f([-1,3]) = \underline{\underline{[-3,3]}}$



$f^{-1} : [-3,3] \to [-1,3]$

iii) $(f \circ g)(x) \underset{\substack{\uparrow \\ \forall x \in \mathbb{R}, \\ \text{poiché } g(\mathbb{R}) = \{-1,1\} \subset \operatorname{dom} f}}{=} f(g(x)) = \begin{cases} 2\log_2 2 & \text{se } x < 0 \\ 3\sqrt[3]{-1} & \text{se } x \geq 0 \end{cases} = \begin{cases} 2\log_2 2 & \text{se } x < 0 \\ -3 & \text{se } x \geq 0. \end{cases}$

$\underset{\forall x \in [-1,3]}{(g \circ f)(x) =} \begin{cases} 1 & \text{se } -1 \leq x < 0 \\ -1 & \text{se } \ 0 \leq x \leq 3. \end{cases}$ □

iv) Per esempio $A = \{-1, 1\}$. A non è unico; si può prendere un qualunque $A = \{x_1, x_2\}$ con $x_1 < 0$ e $x_2 \geq 0$. ■

3) i) $\lim_{x \to -\infty} \dfrac{2^x - x^k}{3 + x^2 + \log|x|} = \begin{cases} 0 & \text{se } k = 1 \\ -1 & \text{se } k = 2 \\ +\infty & \text{se } k = 3. \end{cases}$ □

ii) $\lim_{x \to -1^-} \dfrac{\overset{\to e^{-1}}{e^x}}{\underbrace{x^2 - 1}_{\text{infinitesimo positivo}}} = \underline{+\infty}$

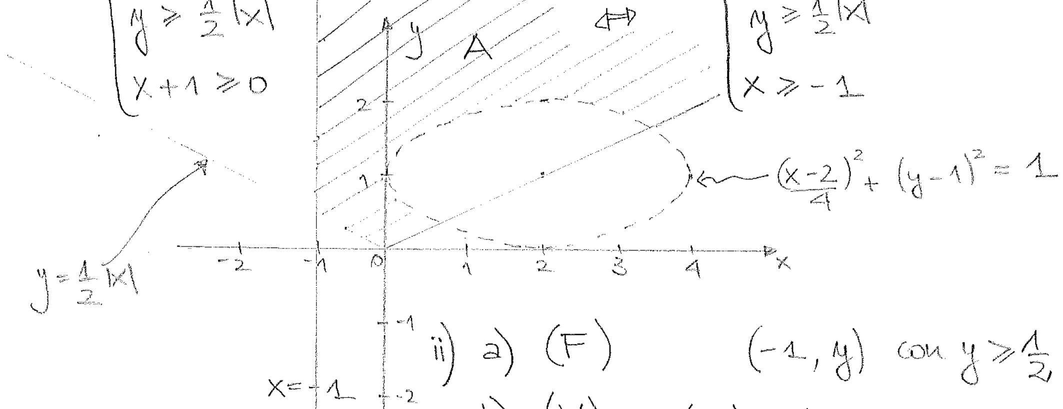
$\lim_{x \to -1^+} \dfrac{\overset{\to e^{-1}}{e^x}}{\underbrace{x^2 - 1}_{\text{infinitesimo negativo}}} = \underline{\underline{-\infty}}.$ □

iii) $f(x) = \log(2x+1) \qquad f^{(1)}(x) = \dfrac{2}{2x+1} \qquad f^{(2)}(x) = \dfrac{-4}{(2x+1)^2}$

$f^{(3)}(x) = \dfrac{16}{(2x+1)^3}$

$\sum_{k=1}^{3} f^{(k)}(1) = \dfrac{2}{3} - \dfrac{4}{9} + \dfrac{16}{27} = \dfrac{18 - 12 + 16}{27} = \underline{\underline{\dfrac{22}{27}}}.$ ■

4) i) $\begin{cases} x^2 + 4y^2 - 4x - 8y + 4 > 0 \\ y \geq \frac{1}{2}|x| \\ x + 1 \geq 0 \end{cases} \iff \begin{cases} (x-2)^2 + 4(y-1)^2 > 4 \\ y \geq \frac{1}{2}|x| \\ x \geq -1 \end{cases}$



□

ii) a) (F) $(-1, y)$ con $y \geq \frac{1}{2}$ sono tutti elementi di A.

b) (V) $(0,0) \in A$.

c) (F) $\forall (x,y) \in A$ si ha $y \geq 0$! ■

5) $\boxed{f(x) = \dfrac{x-1}{x^2-x-2}}$

i) $f(x) = \dfrac{x-1}{(x+1)(x-2)}$ • $\text{dom} f = \mathbb{R}\setminus\{-1,2\}$

• segno di f

$x-1$: $- - - -$ (0 in 1) $+ +$

x^2-x-2: $+ + +$ (0 in -1) $- - -$ (0 in 2) $+ + +$

f: $- - -$ (0 in -1) $+ +$ (0 in 1) $-$ (0 in 2) $+ + +$

• $\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0$

$y=0$ asintoto orizzontale per f per $x\to-\infty$ (per $x\to+\infty$).

$\lim_{x\to-1^-} f(x) = -\infty$ $\quad$ $x=-1$ asintoto verticale per f

$\lim_{x\to-1^+} f(x) = +\infty$ $\quad$ (da sinistra, da destra)

$\lim_{x\to 2^-} f(x) = -\infty$ $\quad$ $x=2$ asintoto verticale per f

$\lim_{x\to 2^+} f(x) = +\infty$ $\quad$ (da sinistra, da destra)

• $\text{dom} f' = \mathbb{R}\setminus\{-1,2\}$ $\quad$ $f'(x) = \dfrac{x^2-x-2-(x-1)(2x-1)}{(x^2-x-2)^2} = \dfrac{x^2-x-2-2x^2+3x-1}{(x^2-x-2)^2}$

$= \dfrac{-x^2+2x-3}{(x^2-x-2)^2} = \dfrac{-(x^2-2x+3)}{(x^2-x-2)^2}$

$\nexists$ sono pt. critici per f

(né di min, né di max)

f': $- - -$ (0 in -1) $- -$ (0 in 2) $- - - -$

f: $\searrow$ $\searrow\ \searrow$ $\searrow$

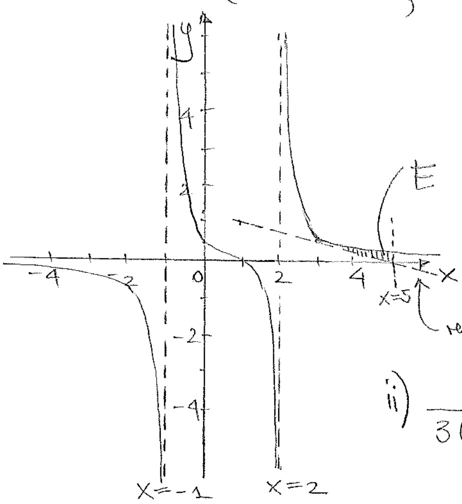


grafico qualitativo di f. $\square$

ii) $\dfrac{1}{3(x-2)} + \dfrac{2}{3(x+1)} = \dfrac{x+1+2(x-2)}{3(x+1)(x-2)}$

$= \dfrac{3x-3}{3(x+1)(x-2)} = \dfrac{\cancel{3}(x-1)}{\cancel{3}(x+1)(x-2)} = f(x)$ $\quad \forall x \in \text{dom} f.$ $\square$

iii) $y = f(3) + f'(3)(x-3)$; ora $f(3) = \frac{1}{2}$, $f'(3) = -\frac{3}{8}$

quindi $y = \frac{1}{2} - \frac{3}{8}(x-3)$, ovvia $y = -\frac{3}{8}x + \frac{13}{8}$ è l'eq. della retta tg al grafico di f nel pt $(3, \frac{1}{2})$.

iv) $$\text{area } E = \int_3^5 \left[f(x) - \left(-\frac{3}{8}x + \frac{13}{8}\right)\right] dx =$$

$$= \int_3^5 f(x)\,dx + \int_3^5 \left(\frac{3}{8}x - \frac{13}{8}\right) dx =$$

$$\underset{\substack{\uparrow \\ \text{usando ii)}}}{=} \frac{1}{3}\int_3^5 \frac{1}{x-2}\,dx + \frac{2}{3}\int_3^5 \frac{1}{x+1}\,dx + \int_3^5 \left(\frac{3}{8}x - \frac{13}{8}\right) dx$$

$$= \frac{1}{3}\Big[\log|x-2|\Big]_3^5 + \frac{2}{3}\Big[\log|x+1|\Big]_3^5 + \left[\frac{3}{8}\frac{x^2}{2} - \frac{13}{8}x\right]_3^5$$

$$= \frac{1}{3}\left[\log 3 - \cancel{\log 1}\right] + \frac{2}{3}\left[\log 6 - \log 4\right] + \left(\frac{3}{8}\cdot\frac{25}{2} - \frac{13}{8}\cdot 5\right) -$$
$$- \left(\frac{3}{8}\cdot\frac{9}{2} - \frac{13}{8}\cdot 3\right)$$

$$= \frac{1}{3}\log 3 + \frac{2}{3}\log\frac{3}{2} + \frac{48}{16} - \frac{26}{8} = \underline{\underline{\log 3 - \frac{2}{3}\log 2 - \frac{1}{4}}}.$$

6) $$C_{8,3}\cdot C_{12,4} = \frac{\cancel{8!}}{5!\,3!}\cdot\frac{12!}{\cancel{8!}\,4!} = \underline{\underline{\frac{12!}{5!\,4!\,3!}}}.$$

FILA B

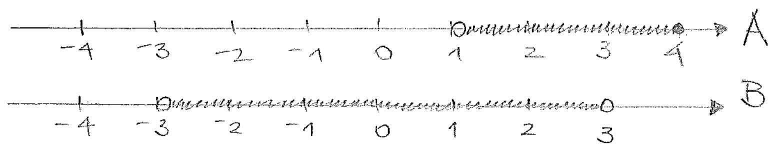
1) i) $A = \{x \in \mathbb{R} : \log_3(x-1) \le 1\}$: $x-1>0$ (altrimenti non è def. $\log_3(x-1)$)
$x-1 \le 3$ (poiché $\log_3 3 = 1$, e usando il fatto che $\log_3 x \uparrow$)

Quindi $\underline{\underline{A =]1,4]}}$.

$B = \left\{x \in \mathbb{R} : \dfrac{|x^2-4|-5}{|x|+3} < 0\right\} = \{x \in \mathbb{R} : |x^2-4| - 5 < 0\}$.

Ora $|x^2-4| < 5 \iff -5 < x^2 - 4 < 5 \iff \underset{\substack{\uparrow \\ \text{sempre}}}{-1 < x^2} < 9 \iff -3 < x < 3$

Quindi $\underline{\underline{B =]-3,3[}}$.

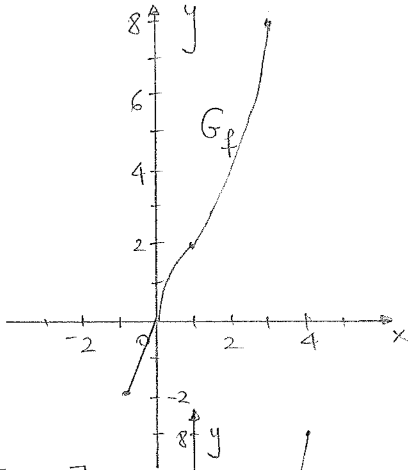


ii) $\underline{\underline{A \cup B =]-3,4]}}$; $\underline{\underline{A \cap B =]1,3[}}$; $\underline{\underline{A \setminus B = [3,4]}}$.

iii) A e B sono insiemi $\underline{\text{limitati}}$; $\underline{\nexists \min A}$; $\underline{\max A = 4}$.

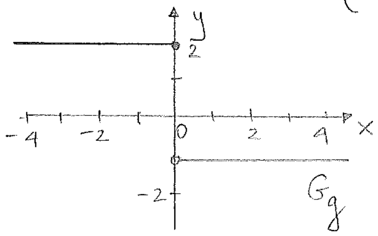
2) i) $f : [-1,3] \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x & \text{se } -1 \le x \le 0 \\ 2\sqrt[3]{x} & \text{se } 0 < x < 1 \\ 2^x & \text{se } 1 \le x \le 3 \end{cases}$$



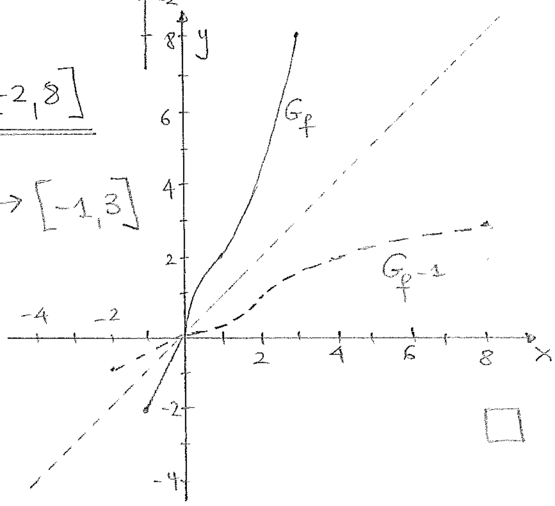
$g : \mathbb{R} \to \mathbb{R}$,

$$g(x) = \begin{cases} 2 & \text{se } x \le 0 \\ -1 & \text{se } x > 0. \end{cases}$$



ii) $f([-1,3]) = \underline{\underline{[-2,8]}}$

$f^{-1} : [-2,8] \to [-1,3]$



iii) $(f \circ g)(x) = f(g(x)) = \begin{cases} 2^2 & \text{se } x \le 0 \\ 2(-1) & \text{se } x > 0 \end{cases}$

$\forall x \in \mathbb{R}$, poiché $g(\mathbb{R}) = \{-1, 2\} \subset \operatorname{dom} f$

$$= \begin{cases} 4 & \text{se } x \le 0 \\ -2 & \text{se } x > 0 \end{cases}.$$

$$(g \circ f)(x) = \begin{cases} 2 & \text{se } -1 \le x \le 0 \\ -1 & \text{se } 0 < x \le 3. \end{cases}$$

$\forall x \in [-1,3]$

iv) Per esempio $A = \{-1, 1\}$. A non è unico; si può prendere un qualunque $A = \{x_1, x_2\}$ con $x_1 \le 0$ e $x_2 > 0$.

3) i) $$\lim_{x \to -\infty} \frac{2^x + x^k}{3 + 2x^2 + \log|x|} = \begin{cases} 0 & \text{se } k = 1 \\ \frac{1}{2} & \text{se } k = 2 \\ -\infty & \text{se } k = 3. \end{cases}$$

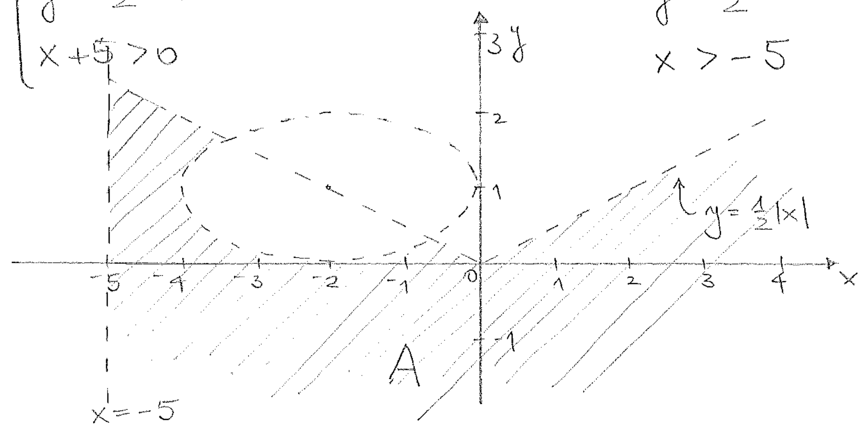
ii) $$\lim_{x \to -2^-} \frac{e^x \to e^{-2}}{\underbrace{x^2 - 4}_{\text{infinitesimo positivo}}} = \underline{\underline{+\infty}}$$

$$\lim_{x \to -2^+} \frac{e^x \to e^{-2}}{\underbrace{x^2 - 4}_{\text{infinitesimo negativo}}} = \underline{\underline{-\infty}}.$$

iii) $f(x) = e^{3x+1}$ $\quad f^{(1)}(x) = 3e^{3x+1}$ $\quad f^{(2)}(x) = 9e^{3x+1}$ $\quad f^{(3)}(x) = 27e^{3x+1}$

$$\sum_{k=1}^{3} f^{(k)}(0) = 3e + 9e + 27e = \underline{\underline{39e}}.$$

4) i) $\begin{cases} x^2+4y^2+4x-8y+4>0 \\ y<\frac{1}{2}|x| \\ x+5>0 \end{cases}$ $\qquad$ $\begin{array}{l} (x+2)^2+4(y-1)^2>4 \\ y<\frac{1}{2}|x| \\ x>-5 \end{array}$



ii) a) (V) poiché $\forall (x,y)\in A$ si ha $x>-5$.

b) (V) per esempio $(1,0)\in A$ e $(x-1)y=0$.

c) (F) prendendo $x=-5$, si ha che $\forall y\in\mathbb{R}$, $(x,y)\notin A$.

5) $\boxed{f(x)=\dfrac{x+1}{x^2+x-2}}$ $\qquad$ i) $f(x)=\dfrac{x+1}{(x+2)(x-1)}$ $\qquad$ • $\text{dom} f=\underline{\mathbb{R}\setminus\{-2,1\}}$

• segno di f

$x+1$: $-$ per $x<-1$, 0 in $x=-1$, $+$ per $x>-1$

x^2+x-2: $+$ per $x<-2$, 0 in $x=-2$, $-$ per $-2<x<1$, 0 in $x=1$, $+$ per $x>1$

f: $-$ per $x<-2$, $+$ per $-2<x<-1$, 0 in $x=-1$, $-$ per $-1<x<1$, $+$ per $x>1$

• $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to+\infty} f(x)=0$

$\underline{y=0 \text{ asintoto orizzontale per } f}$ per $x\to-\infty$ (per $x\to+\infty$).

$\lim\limits_{x\to-2^-} f(x)=-\infty$

$\lim\limits_{x\to-2^+} f(x)=+\infty$ $\qquad$ $\underline{x=-2}$ $\underline{\text{asintoto verticale per } f}$ (da sinistra, da destra)

$\lim\limits_{x\to1^-} f(x)=-\infty$

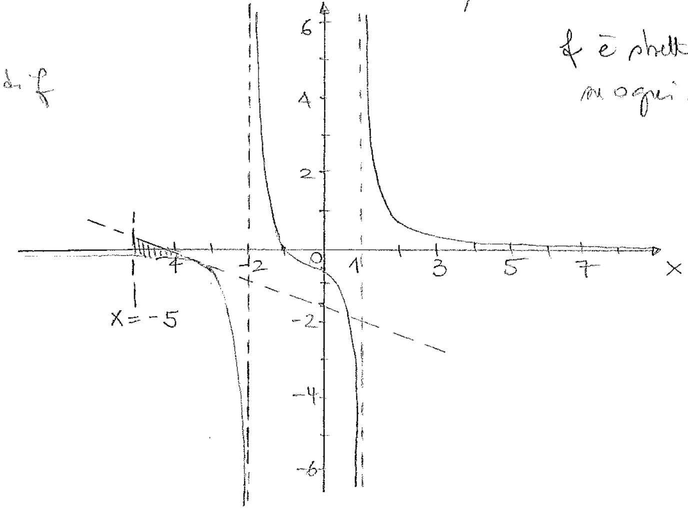
$\lim\limits_{x\to1^+} f(x)=+\infty$ $\qquad$ $\underline{x=1}$ $\underline{\text{asintoto verticale per } f}$ (da sinistra, da destra).

• $\text{dom} f'=\mathbb{R}\setminus\{-2,1\}$ $\qquad$ $f'(x)=\dfrac{x^2+x-2-(x+1)(2x+1)}{(x^2+x-2)^2}=\dfrac{x^2+x-2-2x^2-3x-1}{(x^2+x-2)^2}$

$$f'(x) = \frac{-x^2-2x-3}{(x^2+x-2)^2} = \frac{-(x^2+2x+3)}{(x^2+x-2)^2}$$

∄ sono pt. critici per f ; ∄ sono pt. di massimo/minimo locali per f

grafico qualitativo di f



f è strett. decrescente in ogni sottointervallo.

ii) $\frac{1}{3(x+2)} + \frac{2}{3(x-1)} = \frac{x-1+2(x+2)}{3(x^2+x-2)} = \frac{\cancel{3}(x+1)}{\cancel{3}(x^2+x-2)} = f(x) \quad \forall x \in \operatorname{dom} f.$ $\square$

iii) $y = f(-3) + f'(-3)(x+3)$; ora $f(-3) = -\frac{1}{2}$, $f'(-3) = -\frac{3}{8}$

quindi $y = -\frac{1}{2} - \frac{3}{8}(x+3)$, ossia $y = -\frac{3}{8}x - \frac{13}{8}$ è l'eq. della retta tg al grafico in $\left(-3, -\frac{1}{2}\right)$.

iv) area $E = \int_{-5}^{-3} \left(-\frac{3}{8}x - \frac{13}{8} - f(x)\right) dx =$

$$= \left[-\frac{3}{8}\cdot\frac{x^2}{2} - \frac{13}{8}x\right]_{-5}^{-3} - \int_{-5}^{-3} f(x)\,dx \underset{\text{usando ii)}}{=}$$

$$= \left[\left(-\frac{3}{8}\cdot\frac{9}{2} + \frac{13}{8}\cdot 3\right) - \left(-\frac{3}{8}\cdot\frac{25}{2} + \frac{13\cdot 5}{8}\right)\right] - \frac{1}{3}\int_{-5}^{-3}\frac{1}{x+2}\,dx - \frac{2}{3}\int_{-5}^{-3}\frac{1}{x-1}\,dx$$

$$= \left(\frac{48}{16} - \frac{26}{8}\right) - \frac{1}{3}\Big[\log|x+2|\Big]_{-5}^{-3} - \frac{2}{3}\Big[\log|x-1|\Big]_{-5}^{-3}$$

$$= -\frac{4}{16} - \frac{1}{3}\left[\cancel{\log 1} - \log 3\right] - \frac{2}{3}\left[\log 4 - \log 6\right]$$

$$= \underline{\underline{-\frac{1}{4} + \log 3 - \frac{2}{3}\log 2}}.$$

6) $C_{10,4} \cdot C_{6,3} = \frac{10!}{8!\,4!} \cdot \frac{6!}{3!\,3!} = \frac{10!\,\cancel{6!}}{8\cdot 7\cdot \cancel{6!}\,3!\,3!} = \underline{\underline{\frac{10!}{56\cdot 3!\,3!}}}$