Es. 1

ⓐ $F(u) = \int_a^b \sqrt{2+x^2-\sin u'(x)}\,dx$

$$F(u+\varepsilon v) = \int_a^b \sqrt{2+x^2-\sin(u'+\varepsilon v')}\,dx \qquad v\in X$$

$$\frac{d}{d\varepsilon}F(u+\varepsilon v) = \frac{1}{2}\int_a^b \frac{-\cos(u'+\varepsilon v')\,v'}{\sqrt{2+x^2-\sin(u'+\varepsilon v')}}\,dx$$

$$\delta F(u,v) = \frac{d}{d\varepsilon}F(u+\varepsilon v)\Big|_{\varepsilon=0} = \frac{1}{2}\int_a^b \frac{-(\cos u')\,v'}{\sqrt{2+x^2-\sin u'}}\,dx \quad \forall u,v\in X$$

□

ⓑ $F(u) = \int_a^b \left[x^2u^2 + e^{u'}\right]dx$

$$F(u+\varepsilon v) = \int_a^b \left[x^2(u+\varepsilon v)^2 + e^{u'+\varepsilon v'}\right]dx \qquad v\in X$$

$$\frac{d}{d\varepsilon}F(u+\varepsilon v) = \int_a^b \left[x^2\,2(u+\varepsilon v)\,v + e^{u'+\varepsilon v'}\,v'\right]dx$$

$$\delta F(u,v) = \frac{d}{d\varepsilon}F(u+\varepsilon v)\Big|_{\varepsilon=0} = \int_a^b \left[2x^2uv + e^{u'}v'\right]dx \quad \forall u,v\in X$$

■

Es. 2 . $X = \mathcal{C}^0([a,b])$

$$F(u) = \int_a^b (\sin^3 x + u^2)\,dx$$

$$\delta F(u,v) = \int_a^b 2uv\,dx$$

$$F(u+v) = \int_a^b \left[\sin^3 x + (u+v)^2\right]dx = \int_a^b \left[\sin^3 x + u^2\right]dx + \int_a^b v^2\,dx + 2\int_a^b uv\,dx$$

$$= F(u) + \int_a^b v^2\,dx + \delta F(u,v) \qquad \forall u,v\in X$$

con $F(u+v) = F(u) + \int_a^b v^2 dx + \delta F(u,v) \quad \forall u,v \in X$

che diventa $F(u+v) \geq F(u) + \delta F(u,v) \quad \forall u,v \in X$

e $=$ se e solo se $\int_a^b v^2 dx = 0 \iff v = 0$. ▨

Es. 3 $h \in \mathcal{C}^0([a,b])$ $\int_a^b h(x)u(x)dx = 0 \quad \forall u \in X_0 = \{u \in \mathcal{C}^0 : \int_a^b u = 0\}$

$\Rightarrow h(x) = c$ su $[a,b]$.

Infatti, sia $v \in \mathcal{C}^1([a,b])$ con $v(a) = v(b) = 0$. Poniamo $u(x) = v'(x)$, allora

$$\int_a^b u(x)dx = \int_a^b v'(x)dx = v(b) - v(a) = 0.$$

Quindi tale $u \in X_0$. Allora

$$\int_a^b h(x)\, v'(x)\, dx = \int_a^b h(x)\, u(x)\, dx \overset{\text{per ipotesi}}{=} 0$$

$\Rightarrow \int_a^b h(x)v'(x)dx = 0$. Per l'arbitrarietà di v,

dal lemma di du Bois-Reymond si ottiene $h(x) = c$ su $[a,b]$

▨

Es. 4 $\alpha, \beta \in \mathcal{C}^0([0,b])$ con $\rho > 0$; $X \equiv \mathcal{C}^0([0,b])$

Sia $F(u) = \int_0^b \left[\rho(x)\, u^2(x) + \beta(x)\, u(x)\right] dx$ su X.

ⓐ $F(u) = \int_0^b \left[\rho(x) u^2 + \beta(x) u\right] dx =$

$$= \int_0^b \left(\left[\sqrt{\rho}\, u + \frac{\beta(x)}{2\sqrt{\rho}}\right]^2 - \frac{\beta^2}{4\rho} \right) dx$$

Allora

$$F(u) \geq -\int_0^b \frac{\beta^2}{4\rho}\,dx$$

e $= -\int_0^b \frac{\beta^2}{4\rho}\,dx \iff \sqrt{\rho}\,u + \frac{\beta}{2\sqrt{\rho}} = 0$

ossia $\boxed{u_0(x) = -\frac{\beta(x)}{2\rho(x)}}$

(il funzionale è strettamente convesso).

(b) $\inf_{u \in X} F(u) = -\infty$ se $\rho < 0$; infatti

$$\inf_X F(u) \leq (\max_{[0,b]} \rho(x)) \int_a^b u^2(x)\,dx + \max_{[0,b]} |\beta(x)| \int_0^b |u|\,dx$$

Considerando $u(x) = k$ costante

$$\inf_X F(u) \leq \left[\max_{[0,b]} \rho(x)\right] b\,k^2 + \left[\max_{[0,b]} |\beta(x)|\right] b\,k$$

$\downarrow$

$-\infty$ se $k \to +\infty$. ■

Es. 5

(a) $F(u) = \int_0^1 \left[2e^x u + u'^2\right] dx \qquad X = \{ C^1([0,1]) : u(0) = 0,\ u(1) = 1 \}$

$f(x,u,\xi) = \underbrace{\xi^2}_{\text{strett. convessa in } \xi} + 2e^x u \longrightarrow$ convessa in u

Eq. EL. $\cancel{2}u''(x) = \cancel{2}e^x \qquad u'(x) = e^x + c_1$

$u(x) = e^x + c_1 x + c_2$

$u(0) = 0 \iff 1 + c_2 = 0$

$u(1) = 1 \iff 1 = e + c_1 + c_2 \qquad \Rightarrow \boxed{u(x) = e^x + (2-e)x - 1}$

(b) $F(u) = \int_1^2 \left[2u^2 + x^2 u'^2\right] dx \qquad X = \{u \in \mathcal{C}^1([1,2]) : u(1) = 1,\ u(2) = 5\}$

$f(x,u,\xi) = 2u^2 + x^2\xi^2$ funzione convessa + fun. strett. convessa

Eq. di EE. $\frac{d}{dx}\left[2x^2 u'\right] = 4u$ su $[1,2]$

$$4xu' + 2x^2u'' = 4u$$

$$2x^2u'' + 4xu' - 4u = 0$$

$u(x) = ax^p$

$$2ap(p-1)x^p + 4apx^p - 4ax^p = 0$$

$\Leftrightarrow \quad 2p^2 + 2p - 4 = 0 \quad \Leftrightarrow$

$p^2 + p - 2 = 0 \qquad p_{1/2} = \frac{-1 \pm \sqrt{1+8}}{2} = \begin{cases} 1 \\ -2 \end{cases}$

$u(x) = c_1 x + c_2 \frac{1}{x^2}$.

Le cond. al bordo implicano

$u(1) = c_1 + c_2 = 1$

$u(2) = 2c_1 + \frac{c_2}{4} = 5$ $\Rightarrow$ $c_1 = \frac{19}{7}$, $c_2 = -\frac{12}{7}$

e quindi $\boxed{u(x) = \frac{19}{7}x - \frac{12}{7x^2}}$ ▨

(c) $F(u) = \int_0^{1/2} \left[u + \sqrt{1+u'^2}\right] dx \qquad X = \{u \in \mathcal{C}^1([0,\tfrac{1}{2}]) : u(0) = -1,\ u(\tfrac{1}{2}) = -\tfrac{\sqrt{3}}{2}\}$

$f(x,u,\xi) = u + \sqrt{1+\xi^2}$

↳ convessa ⟶ strettamente convessa

Eq. di EE. $\frac{d}{dx}\left[\frac{u'}{\sqrt{1+u'^2}}\right] = 1$

$$\frac{u'}{\sqrt{1+u'^2}} = x + c_1$$

$$\frac{u'^2}{1+u'^2} = (x+c_1)^2$$

$$u'^2\left(1-(x+c_1)^2\right) = (x+c_1)^2$$

$$u'(x) = \frac{x+c_1}{\sqrt{1-(x+c_1)^2}}$$

$$u(x) = -\sqrt{1-(x+c_1)^2} + c_2$$

Dalle cond. al contorno $\quad -1 = -\sqrt{1-c_1^2} + c_2$
$\quad -\frac{\sqrt{3}}{2} = -\sqrt{1-\left(\frac{1}{2}+c_1\right)^2} + c_2$

$\iff c_1 = c_2 = 0 \implies \boxed{u(x) = -\sqrt{1-x^2}}$

(d) $F(u) = \int_1^8 \left[u'^2 - 4u\right] dx \qquad X = \left\{u \in C^1([1,8]) : u(1) = 2,\ u(8) = -\frac{37}{4}\right\}$

$f(x,u,q) = q^2 - 4u \qquad$ (strett. convessa + lineare)

l'eq. di EE $\quad \frac{d}{dx}[2u'] = -4$

$$\cancel{2}u'' = -\cancel{4}^{2}$$

$u'' = -2 \qquad u'(x) = -2x + c_1$

$$u(x) = -x^2 + c_1 x + c_2$$

$$\left.\begin{aligned} 2 &= -1 + c_1 + c_2 \\ -\frac{37}{4} &= -64 + 8c_1 + c_2 \end{aligned}\right\} \iff \begin{aligned} c_1 + c_2 &= 3 \\ -\frac{37}{4} &= -64 + 8c_1 + c_2 \end{aligned}$$

$$\boxed{u(x) = -x^2 + \frac{207}{28}x - \frac{123}{28}}$$

(e) $F(u)=\int_1^8 \left[u'^4 - 4u\right] dx \qquad X=\left\{u\in C^1([1,8]) : u(1)=2,\ u(8)=-\frac{37}{4}\right\}$

$f(x,u,\xi)=\xi^4-4u$ (strett. convessa + lineare)

~~EE~~ $\frac{d}{dx}\left[4u'^3\right]=-4$

$12u'^2u''=-4$

$3u'^2u''=-1$

$u'=v$

$3v^2v'=-1$

$\left[\frac{v^3}{3}\right]'=-\frac{1}{3}$

$\frac{v^3}{3}=-\frac{1}{3}x+c_1$

$v^3(x)=-x+3c_1$

$v(x)=\sqrt[3]{-x+3c_1}$

$u'(x)=\sqrt[3]{-x+3c_1}$

$u(x)=-\frac{3}{4}\left(-x+3c_1\right)^{4/3}+c_2$

$2=u(1)=-\frac{3}{4}\left(-1+3c_1\right)^{4/3}+c_2$

$-\frac{37}{4}=u(8)=-\frac{3}{4}\left(-8+3c_1\right)^{4/3}+c_2$

$\left(-8+3c_1\right)^{4/3}-\left(-1+3c_1\right)^{4/3}=15$

$c_1=0 \qquad 2+\frac{3}{4}=c_2 \qquad c_2=11/4$

$$\boxed{u(x)=-\frac{3}{4}x^{4/3}+\frac{11}{4}}$$

Es. 6

ⓐ $f(x,u,\xi) = x\xi + u$ è convessa in (u,ξ), ma non strett. convessa in $\mathbb{R}\times\mathbb{R}$ per $x \in [1,2]$ fissato.

$$f(x,v,\eta) \geq f(x,u,\xi) + f_u(x,u,\xi)(v-u) + f_\xi(x,u,\xi)(\eta-\xi)$$

$$x\eta + v \geq x\xi + u + (v-u) + x(\eta-\xi)$$ sempre vera! con $=$

ⓑ $F(u) = \int_1^2 [xu' + u]\,dx$ $\qquad X = \{u \in \mathcal{C}^1([1,2]) : u(1) = 1,\ u(2) = 2\}$

$$F(u) = \int_1^2 [xu' + u]\,dx = \int_1^2 [x\,u]'\,dx$$

$$= [xu]_1^2 = 2u(2) - u(1) = 4 - 1 = 3$$

Allora $F(u) = 3 \quad \forall u \in X$; tutti gli elementi di X minimizzano F.

Es. 7

ⓐ $F(u) = \int_0^1 u'^2\,dx$ $\qquad X_G = \{u \in \mathcal{C}^1([0,1]) : u(0) = 0,\ u(1) = 1,\ G(u) = \int_0^1 x u'\,dx = 1\}$

$f(x,u,\xi) = \xi^2 \qquad g(x,u,\xi) = x\xi$

moltipl. Lagrange $\qquad \frac{d}{dx}\left[2u' + \lambda x\right] = 0$

$2u'' + \lambda = 0 \qquad u'' = -\frac{\lambda}{2}$

$u'(x) = -\frac{\lambda}{2}x + c_1$

$u(x) = -\frac{\lambda}{4}x^2 + c_1 x + c_2$

$u(0)=0 \Leftrightarrow c_2=0$

$u(1)=1 \Leftrightarrow -\frac{\lambda}{4}+c_1=1 \qquad c_1=1+\frac{\lambda}{4}$

$$u(x)=-\frac{\lambda}{4}x^2+\left(1+\frac{\lambda}{4}\right)x$$

$$G(u)=1 \Leftrightarrow \int_0^1 x\left(-\frac{\lambda}{2}x+\left(1+\frac{\lambda}{4}\right)\right)dx=1$$

$$\Leftrightarrow \int_0^1 \left[-\frac{\lambda}{2}x^2+\left(1+\frac{\lambda}{4}\right)x\right]dx=1$$

$$\Leftrightarrow \left[-\frac{\lambda}{2}\frac{x^3}{3}+\left(1+\frac{\lambda}{4}\right)\frac{x^2}{2}\right]_0^1=1$$

$$\Leftrightarrow -\frac{\lambda}{6}+\left(1+\frac{\lambda}{4}\right)\frac{1}{2}=1$$

$$\Leftrightarrow -\frac{\lambda}{6}+\frac{\lambda}{8}=\frac{1}{2} \Leftrightarrow -\frac{\lambda}{3}+\frac{\lambda}{4}=1$$

$$\boxed{\lambda=-12}$$

$$c_1=1-\frac{\cancel{12}^{\,3}}{\cancel{4}}=-2$$

$$\boxed{u(x)=3x^2-2x}$$

(b) $F(u)=\int_0^K u^2\,dx \qquad X_G=\left\{u\in C^1([0,K]) : \cancel{u(0)=0},\ u\geq 0,\ G(u)=\int_0^K (K-x)\,u(x)\,dx=m\right\}$

$f(x,u,\xi)=u^2 \qquad g(x,u,\xi)=(K-x)u$

Eq. EE con multipl. Lagrange $\qquad 0=2u+\lambda(K-x)$

$$u(x)=-\frac{\lambda(K-x)}{2}$$

Dobbiamo trovare $\lambda\leq 0$ in modo che $G(u)=m$

$$m = -\frac{\lambda}{2}\int_0^K (K-x)^2\,dx = -\frac{\lambda}{2}\left[\frac{(K-x)^3}{3}\right]_0^K$$

$$= -\frac{\lambda}{2}\,\frac{K^3}{3}$$

quindi $\lambda = -\frac{6m}{K^3}$ e si ha $\boxed{u(x) = \frac{3m}{K^3}(K-x)}$

la soluzione ricercata.

Es. 8 $F(u) = \int_0^b \sqrt{\cos^2 u + u'^2}\,dx$ $\quad X = \{u \in C^1([0,b]) ;\ u(0) = u(b) = 0\}$, $b > 0$

(a) $\frac{d}{dx}\left[\frac{u'}{\sqrt{\cos^2 u + u'^2}}\right] = \frac{-2\cos u \sin u}{\sqrt{\cos^2 u + u'^2}}$ $\quad f(x,u,\xi) = \sqrt{\cos^2 u + \xi^2}$

i) quindi $u_0(x) \equiv 0$ è una soluzione dell'eq. di EE con $u_0 \in X$ e quindi un estremale di F su X.

ii) $f_\xi(x,u,\xi) = \frac{\xi}{\sqrt{\cos^2 u + \xi^2}}$ $\quad f_{\xi\xi}(x,u,\xi) = \frac{\cos^2 u}{(\cos^2 u + \xi^2)^{3/2}}$

Ora $f_{\xi\xi}(x,0,0) = 1 > 0$ che è la condizione di Legendre stretta per minimizzanti relativi deboli di F su X.

(b) $2Q(v) = \delta^2 F(0,v) = \int_0^b \left[v'^2 - v^2\right] dx$

poiché $f_{\xi\xi}(x,0,0) = 1$

$f_{\xi u}(x,0,0) = 0$

$f_{uu}(x,0,0) = -1$

$$f_u(x,u,\xi) = \frac{-\cos u \sin u}{\sqrt{\cos^2 u + \xi^2}} = \frac{-\frac{1}{2}\sin 2u}{\sqrt{\ }}$$

$$f_{uu}(x,u,\xi) = \frac{-\cos 2u\sqrt{\ } + \frac{1}{2}\sin 2u\,\frac{2\cos u \sin u}{\sqrt{\ }}}{(\cos^2 u + \xi^2)}$$

$$= \frac{\xi^2 \sin^2 u - \xi^2\cos^2 u - \cos^4 u}{(\cos^2 u + \xi^2)^{3/2}}$$

Allora l'eq. (accessoria) di Jacobi è data da

$$v'' + v = 0$$

e $v(x) = c_1 \sin x + c_2 \cos x$, $c_1, c_2 \in \mathbb{R}$ sono i campi di Jacobi relativi ad F ed u_0.

La funzione di Jacobi $\Delta(x,0)$ è l'unica soluzione del probl. di Cauchy

$$\begin{cases} v'' + v = 0 & \text{in } [0,b] \\ v(0) = 0 \\ v'(0) = 1 \end{cases}$$

e si ottiene $v(x) = \sin x$ in $[0,b]$.

(c) Se $b < \pi$, la funzione estremale $u_0 \equiv 0$ è un pt. di minimo relativo debole (stretto) per F su X, poiché non esistono pt. coniugati a 0 in $[0,b]$; se $b > \pi$, l'estremale $u_0 \equiv 0$ non è un pt. di minimo relativo debole, (poiché 0 e $\xi = \pi$ sono una coppia di pt. coniugati in $[0,b]$ e $\xi \in]0,b[$).

<u>Es. 9</u> $F(u) = \int_0^2 [u'^2 - u^2 u'^4]\, dx$ $\quad X^\alpha = \{u \in \mathcal{C}^1([0,2]) : u(0) = a,\ u(2) = a\}$ $a \in \mathbb{R}$.

(a) $f(x,u,\xi) = \xi^2 - u^2 \xi^4$

<u>EE</u> $$\frac{d}{dx}\left[2u' - 4u^2 u'^3\right] = -2u u'^4$$

$$2u'' - 8u u'^4 - 12 u^2 u'^2 u'' = -2u u'^4$$

Ovviamente, per ogni $u_a(x) \equiv a$ è un estremale per F

con X^a.

ⓑ $\mathcal{E}(x,u,\xi,\eta)=$ $\qquad f_\xi(x,u,\xi)=2\xi-4u^2\xi^3$

$$f(x,u,\eta)-f(x,u,\xi)-f_\xi(x,u,\xi)(\eta-\xi)$$
$$=\eta^2-u^2\eta^4-\xi^2+u^2\xi^4-(2\xi-4u^2\xi^3)(\eta-\xi)$$
$$=\eta^2-u^2\eta^4-\xi^2+u^2\xi^4-2\xi\eta+2\xi^2+4u^2\xi^3\eta-4u^2\xi^4$$
$$=\eta^2-u^2\eta^4+\xi^2+u^2\xi^4-2\xi\eta+4u^2\xi^3\eta-4u^2\xi^4$$

$$\mathcal{E}(x,a,0,\eta)=\eta^2-a^2\eta^4=\eta^2(1-a^2\eta^2)\ ;$$

allora se $a=0$, $u_0(x)$ soddisfa la cond. nec. di Weierstraß per minimizzanti relativi forti per F su X^a

se $a\neq 0$, no!

ⓒ $f_\xi(x,u,\xi)=2\xi-4u^2\xi^3$
$f_{\xi\xi}(x,u,\xi)=2-12u^2\xi^2$

$$f_{\xi\xi}(x,a,0)=2>0 \qquad \forall a\in\mathbb{R}\ ;$$

si ha allora che $u_a(x)\equiv a$ soddisfa $\forall a\in\mathbb{R}$ la condizione necessaria di Legendre per minimizzanti relativi deboli per F su X^a.

Es. 10 $\quad F(u) = \int_0^1 \sqrt{1+u'^2(x)}\, dx \qquad X = \{u \in C^1([0,1]) : u(0)=0,\ u(1)=k\}$

$f(x,u,\xi) = \sqrt{1+\xi^2}$

Eq. di EE $\qquad \frac{d}{dx}\left[\frac{u'}{\sqrt{1+u'^2}}\right] = 0$

Allora $u'(x) = k$ è un estremale di F; inoltre $u_0(x) = kx \in X$.

Nota $u_0(x) = kx$ può essere immerso nel campo di estremali $u_\alpha(x) = kx + \alpha \quad \alpha \in \mathbb{R}$ con pendenza $P(x,z) = k \quad (x,z) \in [0,1] \times \mathbb{R}$

$$\mathcal{E}(x,z,k,\eta) = \sqrt{1+\eta^2} - \sqrt{1+k^2} - \frac{k}{\sqrt{1+k^2}}(\eta - k)$$

$$= \sqrt{1+\eta^2} - \frac{1}{\sqrt{1+k^2}} - \frac{k\eta}{\sqrt{1+k^2}}$$

$$= \frac{\sqrt{1+\eta^2}\sqrt{1+k^2} - (1+k\eta)}{\sqrt{1+k^2}}$$

ora $\quad \sqrt{1+\eta^2}\sqrt{1+k^2} \overset{?}{\geq} (1+k\eta) \qquad$ (se $k\eta \leq 0$ non c'è nulla da provare

$\Updownarrow$

$(1+\eta^2)(1+k^2) \overset{?}{\geq} 1 + 2k\eta + \eta^2k^2 \qquad$ se $k\eta > 0$)

$\cancel{1} + \cancel{\eta^2k^2} + \eta^2 + k^2 \overset{?}{\geq} \cancel{1} + 2k\eta + \cancel{\eta^2k^2}$

$\eta^2 + k^2 - 2k\eta \geq 0 \quad ✓$

Quindi $\quad \mathcal{E}(x,z,\underset{\substack{\| \\ P(x,z)}}{k},\eta) \overset{(*)}{\geq} 0 \qquad \forall k,\eta \in \mathbb{R}, \quad \forall z \in \mathbb{R}, \forall x \in [0,1]$

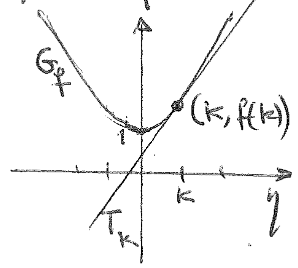
Si risolve il prob. di minimizzare il funzionale della lunghezza della curva congiungente $(0,0)$ con $(1,k)$.

$\blacksquare$

(*) Nota: Si può osservare (come fatto a lezione) che

$$\mathcal{E}(x,z,k,\eta) = f(\eta) - f(k) - f_\eta(k)(\eta-k)$$
$$= f(\eta) - T_k(\eta)$$

dove $f(\eta)=\sqrt{1+\eta^2}$, $T_k(\eta) = f(k) + f_\eta(k)(\eta-k)$ è l'eq. della retta tangente al grafico di f nel pt. $(k, f(k))$.



<u>Es. 11</u> $\quad F(u) = \int_0^b \left[u'^2 + 2uu' - 16u^2\right] dx \qquad X = \{u \in \mathcal{C}^1([0,b]) : u(0)=0,\ u(b)=0\}$

$0 < b < \pi/4$

$$f(x,u,\xi) = \xi^2 + 2u\xi - 16u^2$$

L'eq. di EE $\quad \frac{d}{dx}\left[2u' + 2u\right] = 2u' - 32u$

$$2u'' + \cancel{2u'} = \cancel{2u'} - 32u$$

$$u'' + 16u = 0$$

$$u(x) = c_1 \cos 4x + c_2 \sin 4x \qquad c_1, c_2 \in \mathbb{R}$$

$$u(0)=0 \iff c_1 = 0$$

$$u(b)=0 \iff c_2 = 0 \quad \text{se} \quad 0 < b < \frac{\pi}{4}$$

$u_0(x) \equiv 0$ è soluzione dell'eq. di Eulero-Lagrange (in X)

Osserviamo che $\quad f_\xi(x,u,\xi) = 2\xi + 2u$

$$f_{\xi\xi}(x,u,\xi) = 2 > 0$$

Quindi $f_{\xi\xi}(x,0,0) > 0$ $\quad$ (u_0 soddisfa la cond. di Legendre stretta)

Ma anche $f_{\xi\xi}(x,u,\xi) > 0 \quad \forall (x,u,\xi) \in [0,b] \times \mathbb{R} \times \mathbb{R}$

oss. $\quad f_{\xi\xi}(x,u,\xi) = 2 \qquad f_u = 2\xi - 32u \qquad f_{uu} = -32$

$$f_{\xi u} = 2$$

Quindi $\quad Q(v) = \int_0^b \left[v'^2 + 2vv' - 16v^2\right] dx \quad \forall v \in X$

L'eq. di Eulero associata è dunque nuovamente

$$v'' + 16 v = 0 \qquad \text{e si ha} \quad \Delta(x,0) = \frac{\sin 4x}{4}$$

Se $0 < b < \frac{\pi}{4}$ non ci sono pt. coniugati a 0 in $[0,b]$.

Inoltre

$$\mathcal{E}(x,z,\xi,\eta) = \eta^2 + 2z\eta - \cancel{16z^2} - \xi^2 - 2z\xi + \cancel{16z^2} - (2\xi + 2z)(\eta - \xi)$$
$$= \eta^2 + \cancel{2z\eta} - \xi^2 - \cancel{2z\xi} - 2\eta\xi + 2\xi^2 - \cancel{2z\eta} + \cancel{2z\xi}$$
$$= \eta^2 - 2\eta\xi + \xi^2 = (\eta - \xi)^2 \geq 0$$
$$\forall (x,z,\xi,\eta) \in [a,b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

Quindi possiamo concludere che $u_0 \equiv 0$ è un minimizzante assoluto (stretto) per F su X. □

(b) F non ammette minimo su X se $b > \frac{\pi}{4}$.

Consideriamo, per esempio, $u_k(x) = k \sin \frac{\pi}{b} x \quad k \in \mathbb{N}$;

allora $u_k \in X$ e $u_k'(x) = k \frac{\pi}{b} \cos \frac{\pi}{b} x$. Risulta

$$F(u_k) = \int_0^b \left[k^2 \frac{\pi^2}{b^2} \cos^2 \frac{\pi}{b} x + 2 k^2 \frac{\pi}{b} \sin \frac{\pi}{b} x \cos \frac{\pi}{b} x - 16 k^2 \sin^2 \frac{\pi}{b} x \right] dx$$
$$= k^2 \int_0^b \left[\frac{\pi^2}{b^2} \cos^2 \frac{\pi}{b} x + \frac{\pi}{b} \sin \frac{2\pi}{b} x - 16 \sin^2 \frac{\pi}{b} x \right] dx$$

Poiché $\displaystyle\int_0^b \cos^2 \frac{\pi}{b} x \, dx = \int_0^b \cos \frac{\pi}{b} x \cos \frac{\pi}{b} x \, dx = \cancel{\frac{b}{\pi} \cos \frac{\pi}{b} x \sin \frac{\pi}{b} x \Big|_0^b} + \int_0^b \sin^2 \frac{\pi}{b} x \, dx = \int_0^b \left(1 - \cos^2 \frac{\pi}{b} x\right) dx$

Si ha $\displaystyle\int_0^b \cos^2 \frac{\pi}{b} x \, dx = \frac{b}{2}$; quindi

$$F(u_k) = k^2 \int_0^b \left[\frac{\pi^2}{b^2} \cos^2 \frac{\pi}{b} x + \frac{\pi}{b} \sin \frac{2\pi}{b} x - 16 \sin^2 \frac{\pi}{b} x \right] dx$$

$$= k^2 \left[\frac{\pi^2}{b^{\cancel{2}}} \cdot \frac{\cancel{b}}{2} - 16 \frac{b}{2} \right] = k^2 \left(\frac{\pi^2 - 16 b^2}{2b} \right)$$

Ora se $b > \frac{\pi}{4}$ si ha che $\pi^2 - 16 b^2 < 0$ e quindi $\lim_{k \to +\infty} F(u_k) = -\infty$; risulta allora $\inf_{u \in X} F(u) = -\infty$ e quindi F non ammette minimo su X se $b > \frac{\pi}{4}$. ■