Università di Trento - Dip. di Ingegneria e Scienza dell'Informazione
CdL in Informatica, Ingegneria dell'informazione e delle comunicazioni e Ingegneria dell'informazione e organizzazione d'impresa
a.a. 2017-2018 - PIAZZA 8 - "... non c'è limite - parte 2..."

1) $f, g: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \sin\left(\frac{\pi}{2}x\right) + x e^{\alpha x} & \text{se } x \le 1 \\ \operatorname{arctg}\left(\frac{1}{x-1}\right) & \text{se } x > 1 \end{cases}$$

Poiché $\sin\left(\frac{\pi}{2}x\right) + x e^{\alpha x}$ è continua su $]-\infty, 1]$, basta richiedere che $f(1) = \lim_{x \to 1^+} \operatorname{arctg}\left(\frac{1}{x-1}\right)$, ossia

$\sin\left(\frac{\pi}{2}\right) + e^{\alpha} = \frac{\pi}{2}$, da cui $e^{\alpha} = \frac{\pi}{2} - 1$. Ne segue che $\underline{\alpha = \log\left(\frac{\pi}{2} - 1\right)}$ garantisce la continuità di f in $x = 1$. □

$$g(x) = \begin{cases} \dfrac{\log(1 + x^{\beta})}{\sin 2\sqrt{x}} & \text{se } x > 0 \\ \frac{1}{2}\cos x & \text{se } x \le 0 \end{cases}$$

Poiché $\cos x$ è continua su $[0, 1[$, basta richiedere che

$f(0) = \lim_{x \to 0^+} \dfrac{\log(1 + x^{\beta})}{\sin 2x}$, ossia

$$\frac{1}{2}\cos 0 = \lim_{x \to 0^+} \underbrace{\frac{\log(1 + x^{\beta})}{x^{\beta}}}_{\to 1} \cdot \frac{x^{\beta}}{\sin 2\sqrt{x}} \cdot \underbrace{\frac{2\sqrt{x}}{2\sqrt{x}}}_{\to 1} \iff \beta = \frac{1}{2}.$$

Ne segue che $\beta = \frac{1}{2}$ garantisce la continuità di g in $x = 0$. ■

2) i) $$\lim_{x \to 0} \frac{\log(1 + 3x^2)}{\cos 2\alpha x - 1} = \lim_{x \to 0} \frac{\sin 2x}{\arcsin x} \iff$$

$$\lim_{x \to 0} \underbrace{\frac{\log(1 + 3x^2)}{3x^2}}_{\to 1} \cdot \underbrace{\frac{3x^2}{\cos 2\alpha x - 1} \cdot \frac{(2\alpha x)^2}{(2\alpha x)^2}}_{\to -2} = \lim_{x \to 0} \underbrace{\frac{\sin 2x}{2x}}_{\to 1} \cdot \underbrace{\frac{2x}{\arcsin x}}_{\to 1}$$

$$\iff -\frac{6}{4\alpha^2} = 2 \qquad \nexists\, \alpha \in \mathbb{R} \text{ t.c. i due limiti siano uguali.}$$

□

ii) $$\lim_{n \to +\infty} (\operatorname{arctg} n^{\alpha}) \log(n + e^n) = \begin{cases} +\infty & \text{se } \alpha \ge 0 \\ +\infty & \text{se } -1 < \alpha < 0 \\ 1 & \text{se } \alpha = -1 \\ 0 & \text{se } \alpha < -1 \end{cases}$$

Infatti se $\alpha < 0$

$$(\operatorname{arctg} n^{\alpha}) \log(n + e^n) = \operatorname{arctg}\frac{1}{n^{-\alpha}} \log\left(e^n\left(1 + \frac{n}{e^n}\right)\right) =$$

$$= \left(\operatorname{arctg}\frac{1}{n^{-\alpha}}\right)\left[n + \log\left(1 + \frac{n}{e^n}\right)\right]$$

$$= \underbrace{\left(\frac{\operatorname{arctg}\frac{1}{n^{-\alpha}}}{\frac{1}{n^{-\alpha}}}\right)}_{\to 1} \cdot \frac{1}{n^{-\alpha}}\left[n + \log\left(1 + \frac{n}{e^n}\right)\right] \to \begin{cases} 1 & \text{se } -\alpha = 1 \\ +\infty & \text{se } 0 < -\alpha < 1 \\ 0 & \text{se } -\alpha > 1 \end{cases}$$

$$\lim_{n\to+\infty} n^{\alpha} \arcsin \frac{1}{n^2+1} = \begin{cases} +\infty & \text{se } \alpha>2 \\ 1 & \text{se } \alpha=2 \\ 0 & \text{se } \alpha<2 \end{cases}$$, Infatti

$$n^{\alpha} \arcsin \frac{1}{n^2+1} = \frac{n^{\alpha}}{n^2+1} \cdot \underbrace{\frac{\arcsin \frac{1}{n^2+1}}{\frac{1}{n^2+1}}}_{\to 1} \xrightarrow[n\to+\infty]{} \begin{cases} +\infty & \text{se } \alpha>2 \\ 1 & \text{se } \alpha=2 \\ 0 & \text{se } \alpha<2 \end{cases}$$

$\left[\frac{0}{0}\right]$ 3) i) $\displaystyle\lim_{x\to0} \frac{\sin x^2}{e^{2x^2}-1} = \frac{1}{2}$: $\displaystyle\frac{\sin x^2}{e^{2x^2}-1} = \underbrace{\frac{\sin x^2}{x^2}}_{\to 1} \cdot \underbrace{\frac{2x^2}{e^{2x^2}-1}}_{\to 1} \cdot \frac{1}{2} \xrightarrow[x\to0]{} \frac{1}{2}$

$\left[\frac{0}{0}\right]$ $\displaystyle\lim_{x\to0} \frac{\operatorname{arctg} 3x^2}{\cos x-1} = -6$: $\displaystyle\frac{\operatorname{arctg} 3x^2}{\cos x-1} = \underbrace{\frac{\operatorname{arctg} 3x^2}{3x^2}}_{\to 1} \cdot 3\underbrace{\frac{x^2}{\cos x-1}}_{\to -2} \xrightarrow[x\to0]{} -6$

$\left[\frac{0}{0}\right]$ $\displaystyle\lim_{n\to+\infty} \frac{\arcsin \frac{1}{n}}{\operatorname{arctg} \frac{1}{2n}} = 2$: $\displaystyle\frac{\arcsin \frac{1}{n}}{\operatorname{arctg} \frac{1}{2n}} = \underbrace{\frac{\arcsin \frac{1}{n}}{\frac{1}{n}}}_{\to 1} \cdot \underbrace{\frac{\frac{1}{2n}}{\operatorname{arctg} \frac{1}{2n}}}_{\to 1} \cdot 2 \xrightarrow[n\to+\infty]{} 2$

$[\infty\cdot0]$ ii) $\displaystyle\lim_{x\to+\infty} x^2 \log\left(1+\frac{2}{x^2}\right) = 2$: $\displaystyle x^2 \log\left(1+\frac{2}{x^2}\right) = \underbrace{\frac{\log\left(1+\frac{2}{x^2}\right)}{\frac{2}{x^2}}}_{\to 1} \cdot 2 \xrightarrow[x\to+\infty]{} 2$

$\left[\frac{0}{0}\right]$ $\displaystyle\lim_{x\to0^+} \frac{e^{2\sqrt{x}}-1}{x^2-x+3\sqrt{x}} = \frac{2}{3}$: $\displaystyle\frac{e^{2\sqrt{x}}-1}{x^2-x+3\sqrt{x}} = \frac{e^{2\sqrt{x}}-1}{2\sqrt{x}} \cdot \frac{2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\left(3-\underbrace{\frac{x}{\sqrt{x}}}_{\to 0}+\underbrace{\frac{x^2}{\sqrt{x}}}_{\to 0}\right)} \xrightarrow[x\to0^+]{} \frac{2}{3}$

$\left[\frac{0}{0}\right]$ $\displaystyle\lim_{x\to1} \frac{\sin(x-1)}{x^2-1} = \frac{1}{2}$: $\displaystyle\frac{\sin(x-1)}{x^2-1} = \underbrace{\frac{\sin(x-1)}{(x-1)}}_{\to 1}\frac{1}{(x+1)} \xrightarrow[x\to1]{} \frac{1}{2}$

$\left[\frac{0}{0}\right]$ iii) $\displaystyle\lim_{x\to0} \frac{\operatorname{tg}(\arcsin x)}{2x} = \frac{1}{2}$: $\displaystyle\frac{\operatorname{tg}(\arcsin x)}{2x} = \underbrace{\frac{\operatorname{tg}(\arcsin x)}{\arcsin x}}_{\to 1} \cdot \overbrace{\frac{\arcsin x}{2x}}^{\to 1/2} \xrightarrow[x\to0]{} \frac{1}{2}$

$\left[\frac{0}{0}\right]$ $\displaystyle\lim_{x\to0} \frac{\log(ex+e)-\cos x}{\sin x} = 1$: $\displaystyle\frac{\log(ex+e)-\cos x}{\sin x} =$

$$= \frac{\log[e(x+1)]-\cos x}{\sin x} = \frac{\log e+\log(x+1)-\cos x}{\sin x}$$

$$= \frac{1-\cos x}{\sin x} + \frac{\log(1+x)}{\sin x} = \underbrace{\frac{1-\cos x}{x^2}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{x}{\sin x}}_{\to 1} \cdot \overset{\to 0}{x}$$

$$+ \frac{\log(1+x)}{x} \cdot \frac{x}{\sin x} \xrightarrow[x\to0]{} 1$$

$\left[\frac{0}{0}\right]$ $\displaystyle\lim_{x\to1} \frac{\sin \pi x}{x-1} = -\pi$: $\displaystyle\frac{\sin \pi x}{x-1} \underset{y=x-1}{=} \frac{\sin \pi(y+1)}{y} = \frac{\sin \pi y \cos\pi + \cancel{\cos \pi y \sin \pi}}{y}$

$$= -\underbrace{\frac{\sin \pi y}{\pi y}}_{\substack{\to 1 \\ y\to0}} \cdot \pi \xrightarrow[y\to0]{} -\pi$$

$[\infty\cdot 0]$ 4) $\lim_{n\to+\infty} \frac{1}{\log\left[n^3\left(1-\cos\frac{1}{n}\right)\right]} = \underline{\underline{0}}$: $\frac{1}{\log\left[n^3\left(1-\cos\frac{1}{n}\right)\right]} = \frac{1}{\log\left[\frac{n^3}{n^2}\left(\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}\right)\right]} =$ $\searrow \frac{1}{2}$

$= \frac{1}{\log\left(\frac{n}{2}\right)} \xrightarrow[n\to+\infty]{} 0$

$\left[\frac{\infty}{\infty}\right]$ $\lim_{n\to+\infty} \frac{\log n}{\log\left[n^3\left(1-\cos\frac{1}{n}\right)\right]} = \underline{\underline{1}}$: $\frac{\log n}{\log\left[\frac{n^3}{\cancel{n^2}}\frac{\left(1-\cos\frac{1}{n}\right)}{1/n^2}\right]} = \frac{\log n}{\log n + \log\left(\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}\right)}$ $\searrow \frac{1}{2}$

$\xrightarrow[n\to+\infty]{} 1$

$\left[\frac{\infty}{\infty}\right]$ $\lim_{n\to+\infty} \frac{\log\frac{1}{n^3} + \log\sqrt[3]{n}}{2\log\left(n^4+n^2\right)} = \underline{\underline{-\frac{1}{3}}}$: $\frac{\log\frac{1}{n^3} + \log\sqrt[3]{n}}{2\log\left(n^4+n^2\right)} = \frac{-3\log n + \frac{1}{3}\log n}{2\log\left[n^4\left(1+\frac{1}{n^2}\right)\right]}$

$= \frac{-\frac{8}{3}\log n}{2\log n^4 + \underbrace{2\log\left(1+\frac{1}{n^2}\right)}_{\downarrow 0}} \xrightarrow[n\to+\infty]{} \frac{-\frac{8}{3}}{8} = -\frac{1}{3}$ ■

$\left[\frac{\infty}{\infty}\right]$ 5) i) $\lim_{x\to+\infty} \frac{3x^2 + e^{-x}}{2x^2 + \log x^2} = \underline{\underline{\frac{3}{2}}}$: $\frac{3x^2 + e^{-x}}{2x^2 + \log x^2} \xrightarrow[x\to+\infty]{} \frac{3}{2}$. $\quad (e^{-x} \to 0)$

$\left[\frac{\infty}{\infty}\right]$ $\lim_{x\to+\infty} \frac{3x+e^x}{x^{100}+4^x} = \underline{\underline{0}}$: $\frac{3x+e^x}{x^{100}+4^x} = \frac{e^x\left(\frac{3x}{e^x}+1\right)}{4^x\left(1+\frac{x^{100}}{4^x}\right)} \xrightarrow[x\to+\infty]{} 0$ $\quad \left(\frac{3x}{e^x}\to 0,\ \frac{x^{100}}{4^x}\to 0\right)$

$\left[\frac{\infty}{\infty}\right]$ $\lim_{x\to-\infty} \frac{x^4+\log x^2}{x^2-x} = +\infty$: $\frac{x^4+\log x^2}{x^2-x} =$

$= \frac{x^{\cancel{4}\,2}\left(1+\frac{\log x^2}{x^4}\right)}{\cancel{x^2}\left(1-\frac{1}{x}\right)} \xrightarrow[x\to-\infty]{} +\infty$ $\quad \left(\frac{\log x^2}{x^4}\to 0,\ \frac{1}{x}\to 0\right)$ □

ii) $\lim_{x\to+\infty} \frac{\sin x+3}{\log x - \frac{1}{x}} = \underline{\underline{0}}$: $\frac{\sin x + 3}{\log x - \frac{1}{x}}$ limitato $\xrightarrow[x\to+\infty]{} 0$. $\quad \left(\frac{1}{x}\to 0\right)$

$\left[\frac{\infty}{\infty}\right]$ $\lim_{n\to+\infty} \frac{\log_3\left(n^4+2\cos n\right)}{\log_2\left(n^6+n^2\right)} = \underline{\underline{\frac{2}{3\log_2 3}}}$: $\frac{\log_3\left(n^4+2\cos n\right)}{\log_2\left(n^6+n^2\right)} = \frac{\cancel{4}^{\,2}\log_3 n + \log_3\left(1+\frac{2\cos n}{n^4}\right)}{\cancel{6}_{\,3}\log_2 n + \log_2\left(1+\frac{1}{n^4}\right)}$ $\quad \left(\log_3 n = \frac{\log_2 n}{\log_2 3}\right)$ $\quad \left(\log_3\left(1+\frac{2\cos n}{n^4}\right)\to 0,\ \log_2\left(1+\frac{1}{n^4}\right)\to 0\right)$

$\longrightarrow \underline{\underline{\frac{2}{3\log_2 3}}}$

$\left[\frac{\infty-\infty}{\infty}\right]$ $\lim_{n\to+\infty} \frac{\log\left(2^n+n^2\right)-n}{\log\left(3^n+4n^2\right)+n} = \underline{\underline{\frac{\log 2 - 1}{\log 3 + 1}}}$: $\frac{\log\left(2^n+n^2\right)-n}{\log\left(3^n+4n^2\right)+n} = \frac{n\log 2 + \log\left(1+\frac{n^2}{2^n}\right) - n}{n\log 3 + \log\left(1+\frac{4n^2}{3^n}\right)+n}$

$\xrightarrow[n\to+\infty]{} \frac{\log 2 - 1}{\log 3 + 1}$ ■

$[1^\infty]$ 6) i) $\lim_{n\to+\infty}\left(1+\frac{1}{2n}\right)^{3n} = \underline{e^{3/2}}$: $\Big[\underbrace{\left(1+\frac{1}{2n}\right)^{2n}}_{\to e}\Big]^{\frac{3}{2}} \xrightarrow[n\to+\infty]{} e^{3/2}$

$[1^\infty]$ $\lim_{n\to+\infty}\left(1-\frac{1}{n^2}\right)^{n} = \underline{1}$: $\left(1-\frac{1}{n^2}\right)^{n} = \Big[\underbrace{\left(1-\frac{1}{n^2}\right)^{-n^2}}_{\to e}\Big]^{-\frac{1}{n}} \xrightarrow[n\to+\infty]{} e^0 = 1$

$[1^\infty]$ $\lim_{x\to 0}(1-x)^{\frac{1}{2x}} = \underline{\frac{1}{\sqrt{e}}}$: $(1-x)^{\frac{1}{2x}} = \Big[\underbrace{(1-x)^{-\frac{1}{x}}}_{\to e}\Big]^{-\frac{1}{2}} \xrightarrow[x\to 0]{} e^{-1/2} = \frac{1}{\sqrt{e}}$. □

$[\infty^0]$ ii) $\lim_{x\to+\infty}(x+e^x)^{\frac{\sin x}{x}}$ $\quad \underline{\nexists}$: $(x+e^x)^{\frac{\sin x}{x}} = e^{\frac{\sin x}{x}\log(x+e^x)}$
$= e^{\frac{\sin x}{x}\log\left[e^x\left(1+\frac{x}{e^x}\right)\right]} = e^{\frac{\sin x}{\cancel{x}}\cdot \cancel{x} + \overbrace{\frac{\sin x}{x}\log\left(1+\frac{x}{e^x}\right)}^{\to 0}}$

$[\infty^0]$ $\lim_{x\to+\infty}(x+x^2)^{\frac{\sin x}{x}} = \underline{1}$: $(x+x^2)^{\frac{\sin x}{x}} = e^{\overbrace{\frac{\sin x}{x}\log(x+x^2)}^{\to 0}} \xrightarrow[x\to+\infty]{} e^0 = 1$.

$[1^\infty]$ $\lim_{x\to 0^+}(\cos x)^{\frac{1}{2x^2}} = \underline{\frac{1}{\sqrt[4]{e}}}$: $(\cos x)^{\frac{1}{2x^2}} = e^{\frac{1}{2x^2}\log(\cos x)} = e^{\frac{1}{2x^2}\log(1+(\cos x-1))}$
$= e^{\underbrace{\frac{\cos x-1}{2x^2}}_{\to -\frac{1}{2}}\ \underbrace{\frac{\log(1+(\cos x-1))}{\cos x-1}}_{\to 1}} \xrightarrow[x\to 0^+]{} e^{-1/4}$ □

$[\infty^0]$ iii) $\lim_{n\to+\infty}\sqrt[n]{n+3n^2} = \underline{1}$: $n \le n+3n^2 \le 4n^2 \quad \forall n\ge 1$
$\underbrace{\sqrt[n]{n}}_{\to 1} \le \sqrt[n]{n+3n^2} \le \underbrace{\sqrt[n]{4}\left(\sqrt[n]{n}\right)^2}_{\xrightarrow[n\to+\infty]{} 1}$

$\lim_{x\to 0^+}\left(\sqrt[5]{1+x}\right)^{\frac{1}{\sin x}} = \underline{\sqrt[5]{e}}$: $\left(\sqrt[5]{1+x}\right)^{\frac{1}{\sin x}} = e^{\frac{1}{\sin x}\log\sqrt[5]{1+x}} =$
$= e^{\frac{1}{5\sin x}\log(1+x)} = e^{\frac{1}{5}\underbrace{\frac{x}{\sin x}}_{\to 1}\underbrace{\frac{\log(1+x)}{x}}_{\to 1}} \to e^{1/5}$

$[1^\infty]$ $\lim_{x\to 0^+}\left(\sqrt[5]{1+x^2}\right)^{\frac{1}{1-\cos x}} = \underline{\sqrt[5]{e^2}}$: $\left(\sqrt[5]{1+x^2}\right)^{\frac{1}{1-\cos x}} = e^{\underbrace{\frac{x^2}{5(1-\cos x)}}_{\to 2}\frac{\log(1+x^2)}{x^2}} \to e^{\frac{2}{5}}$ ■

7) $\lim_{n\to+\infty} 3n^3\log\left(1+\frac{1}{n^3+\alpha n^\alpha}\right) = \begin{cases} 3 & \text{se } 0\le\alpha<3 \\ \frac{3}{4} & \text{se } \alpha=3 \\ 0 & \text{se } \alpha>3. \end{cases}$ (usare che $\frac{\log(1+x)}{x}\xrightarrow[x\to 0]{} 1$!)

8) $\lim_{n\to+\infty}\left(\frac{2^n+n^n}{n!}\right)^{\frac{1}{n}} = \underline{e}$: Ricordiamo che $n! \sim \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ per $n\to+\infty$

Infatti $\left(\frac{2^n+n^n}{n!}\right)^{1/n} = e^{\frac{1}{n}\log\left(\frac{n^n\left(\frac{2^n}{n^n}+1\right)}{n!}\right)} = e^{\frac{1}{n}\log\left(\frac{\cancel{n^n}\left(\frac{2^n}{n^n}+1\right)e^n}{\cancel{n^n}\sqrt{2\pi n}}\right)}$
$= e^{\frac{1}{n}\left(\log e^n + \log\left(\frac{2^n}{n^n}+1\right) - \log\sqrt{2\pi n}\right)} = e^{\frac{1}{n}\left[n+\log\left(\frac{2^n}{n^n}+1\right)-\log\sqrt{2\pi n}\right]}$
$\xrightarrow[n\to+\infty]{} e$. ■

si può anche procedere scrivendo $f(x)^{g(x)} = e^{g(x)\log f(x)}$ e poi usando limite notevole del logaritmo.