

Università di Trento - Dip. di Ingegneria e Scienza dell'informazione
CdL in Informatica, Ingegneria dell'informazione e delle comunicazioni e
Ingegneria dell'informazione e organizzazione d'impresa
a.a. 2017-2018 - Foglio d'esercizi 14

"il mondo delle primitive - integrali e la funz. integrale - giusto qualche conto"

$$14.1) a) f(x) = \begin{cases} \int_0^x (e^{-t^2} + t) dt & \text{se } x > 0 \\ a^2 \sin x + b & \text{se } x \leq 0. \end{cases}$$

Abbiamo che f è continua e derivabile $\forall x \neq 0$. Dobbiamo solo vedere la continuità e derivabilità in $x=0$.

f è continua in $x=0 \Leftrightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0)$. Per questo deve essere

$b=0$. Notiamo che per il TFC si ha $f'(x) = e^{-x^2} + x$ se $x > 0$.
(abb. fissato ora $b=0$)

D'altra parte $f'(x) = a^2 \cos x$ se $x < 0$. Poiché f è continua, e $f'_+(0) = \lim_{x \rightarrow 0^+} f'(x) = 1$,
 $f'_-(0) = \lim_{x \rightarrow 0^-} f'(x) = a^2$, f è derivabile in $x=0$ se e solo se $a^2=1$,
quindi $a = \pm 1$. □

$$b) \lim_{x \rightarrow 0^+} \frac{2}{x^2} \int_0^{\log^2(1+2x)} \sqrt[3]{1+t^2} dt = \lim_{x \rightarrow 0^+} \frac{2 \int_0^{\log^2(1+2x)} \sqrt[3]{1+t^2} dt}{(x^2)^{\frac{1}{3}}} = \frac{f(x)}{g(x)} = \frac{0}{0}$$

Notiamo che il limite si presenta nella forma $\frac{0}{0}$, inoltre $g'(x) = 2x \neq 0$ in un intorno destro di 0. Oss. che

$$\frac{f'(x)}{g'(x)} = \frac{2 \sqrt[3]{1 + \log^4(1+2x)} \cdot 2 \log(1+2x) \cdot \frac{2}{1+2x}}{2x} \xrightarrow{x \rightarrow 0^+} 8.$$

TFC

Dal teorema di de l'Hôpital segue dunque che $\lim_{x \rightarrow 0^+} \frac{2}{x^2} \int_0^{\log^2(1+2x)} \sqrt[3]{1+t^2} dt = 8$. □

14.2) i) $F(x) = \frac{x}{2} \sqrt{4+x^2} + 2 \log \left(\frac{x}{2} + \frac{1}{2} \sqrt{4+x^2} \right)$ è derivabile su $\mathbb{R}$ e

$$\begin{aligned} F'(x) &= \frac{1}{2} \sqrt{4+x^2} + \frac{x}{2} \cdot \frac{1}{\sqrt{4+x^2}} \cdot \frac{2x}{2} + 2 \left[\frac{\frac{1}{2} + \frac{1}{2} \frac{2x}{\sqrt{4+x^2}}}{\frac{x}{2} + \frac{1}{2} \sqrt{4+x^2}} \right] = \\ &= \frac{1}{2} \sqrt{4+x^2} + \frac{x^2}{2\sqrt{4+x^2}} + 2 \left[\frac{1 + \frac{x}{\sqrt{4+x^2}}}{x + \sqrt{4+x^2}} \right] = \\ &= \frac{1}{2} \sqrt{4+x^2} + \frac{x^2}{2\sqrt{4+x^2}} + 2 \left[\frac{1}{\sqrt{4+x^2}} \right] = \frac{(4+x^2) + x^2 + 4}{2\sqrt{4+x^2}} = \\ &= \frac{2(4+x^2)}{2\sqrt{4+x^2}} = \sqrt{4+x^2} = f(x) \text{ su } \mathbb{R}. \end{aligned}$$

□

$$ii) \int_0^1 f(x) dx = F(1) - F(0) = \underline{\underline{\frac{1}{2}\sqrt{5} + 2 \log\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)}}. \quad \square$$

$$iii) \lim_{x \rightarrow 0} \frac{F(x) - 2x}{x^3} = \frac{0}{0}$$

$$\text{Oss. che } \frac{(F(x) - 2x)'}{(x^3)'} = \frac{F'(x) - 2}{3x^2} = \frac{f(x) - 2}{3x^2} = \frac{\sqrt{4+x^2} - 2}{3x^2} =$$

$$= \frac{(\sqrt{4+x^2} - 2)(\sqrt{4+x^2} + 2)}{3x^2(\sqrt{4+x^2} + 2)} = \frac{\cancel{4} + x^2 - \cancel{4}}{3x^2(\sqrt{4+x^2} + 2)} \xrightarrow{x \rightarrow 0} \frac{1}{12}$$

Per il teorema di de L'Hôpital segue che $\lim_{x \rightarrow 0} \frac{F(x) - 2x}{x^3} = \underline{\underline{\frac{1}{12}}}$ ■

$$14.3) i) f(x) = \cos 3x + e^{-2x} \quad F(x) = \underline{\underline{\frac{\sin 3x}{3} - \frac{e^{-2x}}{2} + c}} \quad c \in \mathbb{R};$$

$$ii) f(x) = \frac{(\log x)^3}{x} - x 2^{x^2} \quad F(x) = \underline{\underline{\frac{(\log x)^4}{4} - \frac{2^{x^2}}{2 \log 2} + c}} \quad c \in \mathbb{R};$$

$$iii) f(x) = \frac{1}{1+4x^2} + \frac{2}{1-x} \quad F(x) = \underline{\underline{\frac{\arctan 2x}{2} - 2 \log |1-x| + c}} \quad c \in \mathbb{R}. \quad \blacksquare$$

$$14.4) \int \frac{\sqrt[3]{x} - \log^2 x}{2x} dx = \frac{1}{2} \int x^{-\frac{2}{3}} dx - \frac{1}{2} \int \frac{\log^2 x}{x} dx = \frac{1}{2} \cdot \frac{x^{\frac{1}{3}}}{\frac{1}{3}} - \frac{1}{2} \cdot \frac{\log^3 x}{3} + c$$

$$= \underline{\underline{\frac{3}{2} x^{\frac{1}{3}} - \frac{1}{6} \log^3 x + c}} \quad c \in \mathbb{R}.$$

$$\int \frac{x}{x^2+1} dx = \underline{\underline{\frac{1}{2} \log(1+x^2) + c}}, \quad c \in \mathbb{R}.$$

$$\int \frac{x}{\sqrt{1-4x^2}} dx = -\frac{1}{8} \int -8x (1-4x^2)^{-\frac{1}{2}} dx = -\frac{1}{8} \frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \quad c \in \mathbb{R}$$

$$= \underline{\underline{-\frac{1}{4} (1-4x^2)^{\frac{1}{2}} + c}} \quad c \in \mathbb{R}.$$

$$(*) \int \frac{1}{\cos^2 2x} dx = \underline{\underline{\frac{1}{2} \tan(2x) + c}}, \quad c \in \mathbb{R}. \quad \blacksquare$$

$$14.5) \int_1^2 \frac{x^3 - 2x + e}{x} dx = \int_1^2 \left(x^2 - 2 + \frac{e}{x}\right) dx = \left[\frac{x^3}{3} - 2x + e \log x\right]_1^2 = \left(\frac{8}{3} - 4 + e \log 2\right) - \left(\frac{1}{3} - 2\right)$$

$$= \underline{\underline{\frac{7}{3} - 2 + e \log 2}} = \underline{\underline{\frac{1}{3} + e \log 2}}.$$

$$\int_0^2 \frac{9^x - 1}{3^x + 1} dx = \int_0^2 \frac{(3^x - 1)(3^x + 1)}{(3^x + 1)} dx = \left[\frac{3^x}{\log 3} - x\right]_0^2 = \left(\frac{9}{\log 3} - 2\right) - \left(\frac{1}{\log 3}\right) = \underline{\underline{\frac{8}{\log 3} - 2}}$$

$$(*) \int \frac{-2}{\sqrt{1-4x^2}} dx = \underline{\underline{-\operatorname{arctan}(2x) + c}}, \quad c \in \mathbb{R}.$$

$$\begin{aligned}
 \int_{-1}^4 |x^2 - 2x| dx &= \int_{-1}^0 (x^2 - 2x) dx + \int_0^4 (-x^2 + 2x) dx \\
 &= \left[\frac{x^3}{3} - x^2 \right]_{-1}^0 + \left[-\frac{x^3}{3} + x^2 \right]_0^4 \\
 &= -\left(-\frac{1}{3} - 1\right) + \left(-\frac{64}{3} + 16\right) = \frac{49}{3} - 7 = \frac{28}{3}
 \end{aligned}$$

$x^2 - 2x = (x-1)^2 - 1$

$$\begin{aligned}
 \int_{-\pi}^{\frac{\pi}{2}} |\sin x| dx &= \int_{-\pi}^0 (-\sin x) dx + \int_0^{\frac{\pi}{2}} \sin x dx = [\cos x]_{-\pi}^0 + [-\cos x]_0^{\frac{\pi}{2}} \\
 &= (1+1) + 1 = 3.
 \end{aligned}$$

14.6) $\int x \arcsin x dx = \frac{x^2}{2} \arcsin x - \int \frac{x^2}{2} \frac{1}{\sqrt{1-x^2}} dx =$

$$\begin{aligned}
 &= \frac{x^2}{2} \arcsin x - \frac{1}{2} \int x \cdot \frac{x}{\sqrt{1-x^2}} dx = \frac{x^2}{2} \arcsin x - \frac{1}{2} \left[x\sqrt{1-x^2} + \int \sqrt{1-x^2} dx \right] \\
 &\stackrel{(*)}{=} \frac{x^2}{2} \arcsin x - \frac{1}{2} \left[x\sqrt{1-x^2} + \frac{x\sqrt{1-x^2} + \arcsin x}{2} \right] + c
 \end{aligned}$$

$(*)$ $\int \sqrt{1-x^2} dx \stackrel{\uparrow}{=} \int \cos^2 t dt = \int \cos t \cos t dt =$
 $\begin{aligned} &= \sin t \cos t + \int \sin^2 t dt \\ &= \sin t \cos t + t - \int \cos^2 t dt \end{aligned}$
 $\Rightarrow \int \cos^2 t dt = \frac{\sin t \cos t + t}{2} =$
 $= \frac{x\sqrt{1-x^2} + \arcsin x}{2}$

$(**) = \left(\frac{x^2}{2} - \frac{1}{4}\right) \arcsin x + \frac{1}{4} x\sqrt{1-x^2} + c$
 $= \frac{1}{4} (x\sqrt{1-x^2} + (2x^2-1) \arcsin x) + c, c \in \mathbb{R}. \quad \square$

$$\begin{aligned}
 \int x^4 \log 2x dx &= \frac{x^5}{5} \log 2x - \int \frac{x^5}{5} \cdot \frac{1}{x} \cdot 2 dx \\
 &= \frac{x^5}{5} \log 2x - \frac{1}{5} \frac{x^5}{5} + c = \frac{x^5}{5} \left(\log 2x - \frac{1}{5} \right) + c \quad c \in \mathbb{R}. \quad \square
 \end{aligned}$$

$$\begin{aligned}
 \int (2x+1)^2 e^x dx &= \int (4x^2 + 4x + 1) e^x dx = 4 \int x^2 e^x dx + 4 \int x e^x dx + \int e^x dx \\
 &= 4 \left[x^2 e^x - \int 2x e^x dx \right] + 4 \int x e^x dx + e^x + c \\
 &= 4x^2 e^x - 4 \int x e^x dx + e^x + c = 4x^2 e^x - 4 \left[x e^x - \int e^x dx \right] \\
 &\quad + e^x + c = \underline{e^x (4x^2 - 4x + 5) + c} \quad c \in \mathbb{R}. \quad \square
 \end{aligned}$$

$$\begin{aligned}\int x^3 e^{-x} dx &= -x^3 e^{-x} + \int 3x^2 e^{-x} dx = -x^3 e^{-x} - 3x^2 e^{-x} + \int 6x e^{-x} dx \\ &= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} + 6 \int e^{-x} dx \\ &= \underline{\underline{e^{-x} [-x^3 - 3x^2 - 6x - 6] + c}}, \quad c \in \mathbb{R}.\end{aligned}$$

$$\begin{aligned}\int \log(1+x^2) dx &= x \log(1+x^2) - \int x \cdot \frac{2x}{1+x^2} dx = x \log(1+x^2) - 2 \int \frac{x^2}{1+x^2} dx \\ &= x \log(1+x^2) - 2 \int \frac{x^2+1}{x^2+1} dx + 2 \int \frac{1}{x^2+1} dx \\ &= \underline{\underline{x \log(1+x^2) - 2x + 2 \arctan x + c}}, \quad c \in \mathbb{R}.\end{aligned}$$

14.7) $\int \frac{x+1}{\sqrt{x}+1} dx = \int \frac{t^2+1}{t+1} \cdot 2t dt = 2 \int \frac{t^3+t}{t+1} dt = 2 \int (t^2-t+2) dt - 4 \int \frac{1}{t+1} dt$

$\begin{matrix} \uparrow \\ \sqrt{x}=t \\ x=t^2 \\ dx=2t dt \end{matrix}$

$$\begin{array}{r} t^3+t : t+1 = t^2-t+2 \\ \underline{t^3+t^2} \\ -t^2+t \\ \underline{+t} \\ 2t \\ \underline{-2t-2} \\ -2 \end{array}$$

$$\begin{aligned}&= 2 \left[\frac{t^3}{3} - \frac{t^2}{2} + 2t \right] - 4 \log(1+t) + c \\ &= \underline{\underline{2 \left[\frac{(\sqrt{x})^3}{3} - \frac{x}{2} + 2\sqrt{x} \right] - 4 \log(1+\sqrt{x}) + c}}, \quad c \in \mathbb{R}.\end{aligned}$$

$$\begin{aligned}\int \sqrt{4-x^2} dx &= \int \sqrt{4-4\sin^2 t} \cdot 2 \cos t dt = 4 \int \cos^2 t dt = \\ &\stackrel{X=2\sin t}{dx=2\cos t} \stackrel{(14.6)}{=} 2 \int (\sin t \cos t + t) dt + c = \\ &= \underline{\underline{x \sqrt{4-x^2} + 2 \arcsin \frac{x}{2} + c}}, \quad c \in \mathbb{R}.\end{aligned}$$

$$\int \frac{e^x}{e^{2x}+1} dx = \underline{\underline{\arctan e^x + c}}, \quad c \in \mathbb{R}.$$

$$\begin{aligned}\int \frac{1}{\cos x} dx &\stackrel{\uparrow}{=} \int \frac{1+t^2}{1-t^2} \cdot \frac{2}{1+t^2} dt = 2 \int \left[\frac{1}{1-t} + \frac{1}{1+t} \right] dt \\ &\stackrel{t=\tanh \frac{x}{2}}{\substack{X=2\operatorname{arctg} t \\ dx=\frac{2}{1+t^2} dt}} \quad \cos t = \frac{1-t^2}{1+t^2} \quad = -\log|1-t| + \log|1+t| + c \\ &= \log \left| \frac{1+t}{1-t} \right| + c \\ &= \underline{\underline{\log \left| \frac{1+\tanh \frac{x}{2}}{1-\tanh \frac{x}{2}} \right| + c}}\end{aligned}$$

$$\begin{aligned}\int \sqrt{2+x^2} dx &\stackrel{\uparrow}{=} \int \sqrt{2+2\sinh^2 t} \cdot \sqrt{2} \cosh t dt = 2 \int \cosh^2 t dt \\ &\stackrel{X=\sqrt{2}\sinh t}{dx=\sqrt{2}\cosh t} \quad\end{aligned}$$

$$\begin{aligned} \text{On } \int \cosh^2 t dt &= \cosh t \sinh t - \int \sinh^2 t dt = \cosh t \sinh t + \int (1 - \cosh^2 t) dt \\ \Rightarrow \int \cosh^2 t dt &= \frac{\cosh t \sinh t + t}{2} + c \end{aligned}$$

$$\begin{aligned} \text{Donc } \int \sqrt{2+x^2} dx &= \left[\frac{x}{\sqrt{2}} \sqrt{1+\frac{x^2}{2}} + \sinh^{-1}\left(\frac{x}{\sqrt{2}}\right) \right] + c \\ &= \frac{x}{2} \sqrt{2+x^2} + \log\left(\frac{x}{\sqrt{2}} + \sqrt{1+\frac{x^2}{2}}\right) + c, \quad c \in \mathbb{R}. \end{aligned}$$

$$\begin{aligned} 14.8) \int \frac{1}{2x-4x^2} dx &= \frac{1}{2} \int \frac{1}{x-2x^2} dx = \frac{1}{2} \int \frac{1}{x(1-2x)} dx = \\ &= \frac{1}{2} \left[\int \frac{2}{1-2x} dx + \int \frac{1}{x} dx \right] = -\frac{1}{2} \log|1-2x| + \frac{1}{2} \log|x| + c \\ &= \frac{1}{2} \log\left|\frac{x}{1-2x}\right| + c, \quad c \in \mathbb{R}. \end{aligned}$$

$$\begin{aligned} \int \frac{x+3}{x^2+2} dx &= \int \frac{x}{x^2+2} dx + \frac{1}{2} \int \frac{3}{\left(\frac{x}{\sqrt{2}}\right)^2+1} dx \\ &= \frac{1}{2} \int \frac{2x}{x^2+2} dx + \frac{1}{\sqrt{2}} \cdot 3 \int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1} dx \\ &= \frac{1}{2} \log(x^2+2) + \frac{3}{\sqrt{2}} \operatorname{arctg}\left(\frac{x}{\sqrt{2}}\right) + c, \quad c \in \mathbb{R}. \end{aligned}$$

$$\begin{aligned} \int \frac{x+1}{x(x-2)} dx &= -\frac{1}{2} \int \frac{1}{x} dx + \frac{3}{2} \int \frac{1}{x-2} dx \\ &= -\frac{1}{2} \log|x| + \frac{3}{2} \log|x-2| + c, \quad c \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} \int \frac{3x^3+x^2}{x+1} dx &= \int \left(3x^2-2x+2 - \frac{2}{x+1}\right) dx \\ &= \underline{3x^3 - x^2 + 2x - 2 \log|x+1| + c} \end{aligned}$$

$$\begin{array}{r} 3x^3+x^2 : x+1 = 3x^2-2x+2 \\ \underline{3x^3+3x^2} \\ -2x^2 \\ \underline{+2x^2+2x} \\ 2x \\ \underline{-2x+2} \\ 2 \end{array}$$

$$\begin{aligned} \int \frac{1}{x^2+x+2} dx &= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{7}{4}} dx \\ &= \frac{4}{7} \int \frac{1}{\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{7}}{2}}\right)^2 + 1} dx = \frac{2}{\sqrt{7}} \int \frac{\frac{\sqrt{7}}{2}}{\left(\frac{\sqrt{7}}{2}\left(x+\frac{1}{2}\right)\right)^2 + 1} dx \\ &= \underline{\frac{2}{\sqrt{7}} \operatorname{arctg}\left(\frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right)\right) + c}, \quad c \in \mathbb{R}. \end{aligned}$$

14.9) $f(x) = -xe^{x^2} + e$

$\bullet \text{ dom } f = \mathbb{R}$

$\bullet \lim_{x \rightarrow -\infty} f(x) = +\infty$

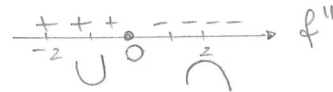
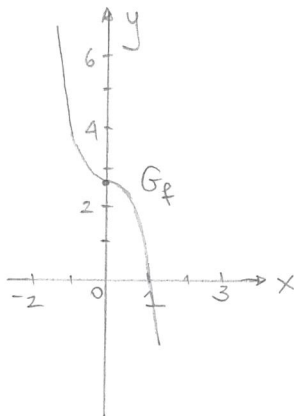
$\bullet \lim_{x \rightarrow +\infty} f(x) = -\infty$

$\bullet f$ segno



$\bullet f'(x) = -e^{x^2} - 2x^2 e^{x^2} = -e^{x^2}(1 + 2x^2) < 0 \quad f \downarrow$

$\bullet f''(x) = -2xe^{x^2} - 4xe^{x^2} - 4x^3 e^{x^2} = -2xe^{x^2}(3 + 2x^2)$



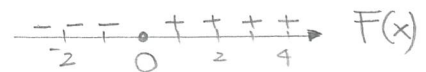
area E = $\int_0^1 f(x) dx =$

$$= - \int_0^1 (xe^{x^2} + e) dx = \left[-\frac{1}{2} e^{x^2} + ex \right]_0^1$$

$$= \left(-\frac{1}{2} e + e \right) - \left(-\frac{1}{2} \right) = \underline{\underline{\frac{e}{2} + \frac{1}{2}}}$$

□

14.10) $F(x) = \int_0^x \underbrace{(e^{2t^2} - 4e^{t^2} + 5)}_{f(t) > 0} dt$

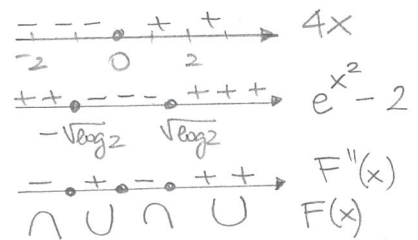


$F'(x) = e^{2x^2} - 4e^{x^2} + 5 > 0 \quad \forall x \in \mathbb{R}$

$F \nearrow$ su $\mathbb{R}$

$$F''(x) = 4xe^{2x^2} - 8xe^{x^2}$$

$$= 4xe^{x^2} [e^{x^2} - 2]$$



$x^2 = \log 2$
 $x = \pm \sqrt{\log 2}$

F è concava su $]-\infty, -\sqrt{\log 2}]$, in $[0, \sqrt{\log 2}]$; è convessa in $[-\sqrt{\log 2}, 0]$ e in $[\sqrt{\log 2}, +\infty[$. I pt. $x = \pm \sqrt{\log 2}$, $x = 0$ sono pt. di flesso.

■