Università di Trento - Dip. di Ingegneria e Scienza dell'Informazione
CdL in Informatica, Ingegneria dell'informazione e delle comunicazioni e
Ingegneria dell'informazione e organizzazione d'impresa
a.a. 2017-18 — PIAZZA 3 — "Primi passi in $\mathbb{C}$."

1.1) Scrivete in forma algebrica i seguenti numeri complessi:

a) $(2i+3)(1-i) \underset{\substack{\uparrow \\ i^2=-1}}{=} 2i + 2 + 3 - 3i = \underline{\underline{5-i}}.$

b) $\dfrac{i}{1+i} = \dfrac{i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{i+1}{2} = \underline{\underline{\dfrac{1}{2} + \dfrac{1}{2} i}}.$

c) $\dfrac{(1+2i)^2}{1-i} = \dfrac{(1+2i)^2}{1-i} \cdot \dfrac{1+i}{1+i} = \dfrac{(1+4i-4)(1+i)}{2} =$

$= \dfrac{(-3+4i)(1+i)}{2} = \dfrac{-3-3i+4i-4}{2} = \underline{\underline{-\dfrac{7}{2} + \dfrac{i}{2}}}.$

1.2) a) $\overline{3i(1+2i)} = \overline{3i-6} = \overline{-6+3i} = \underline{\underline{-6-3i}}.$

b) $\overline{\dfrac{1}{1-i} + \dfrac{3}{2i}} = \overline{\dfrac{1+i}{(1-i)(1+i)} + \dfrac{3}{2i}\dfrac{(-2i)}{(-2i)}} = \overline{\dfrac{1+i}{2} - \dfrac{\cancel{6}^{3} i}{\cancel{4}_{2}}}$

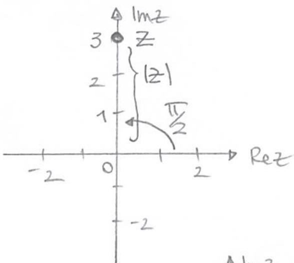
$= \overline{\dfrac{1}{2} - i} = \underline{\underline{\dfrac{1}{2} + i}}$

1.3) $z = 2i$ $\quad \mathrm{Re}\big((z+1)(\overline{z}+3)\big) = \mathrm{Re}\big((2i+1)(-2i+3)\big) =$

$= \mathrm{Re}\big(4+6i-2i+3\big) = \mathrm{Re}\big(7+4i\big) = \underline{\underline{7}}$

$\mathrm{Im}\big(|z|i + \overline{(z+1)}\big) = \mathrm{Im}\big(2i + \overline{(2i+1)}\big) = \mathrm{Im}\big(\cancel{2i} + 1 - \cancel{2i}\big) = \underline{\underline{0}}$

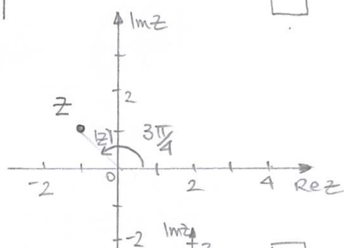
1.4) a) $z = 3i$ $\qquad \underline{\underline{z = 3\left(\cos\dfrac{\pi}{2} + i \sin\dfrac{\pi}{2}\right)}}.$



b) $z = i(1+i) = -1 + i$

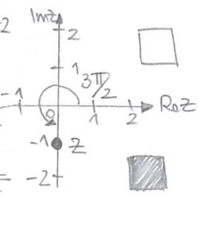
$z = \sqrt{2}\left(-\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} i\right)$

$\underline{\underline{z = \sqrt{2}\left(\cos\dfrac{3\pi}{4} + i \sin\dfrac{3\pi}{4}\right)}}.$

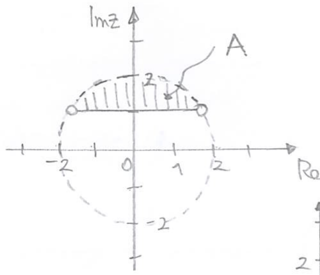


c) $z = \dfrac{1-i}{1+i} = \dfrac{1-i}{1+i} \cdot \dfrac{1-i}{1-i} = \dfrac{\cancel{1} - 2i - \cancel{1}}{2} = -i$

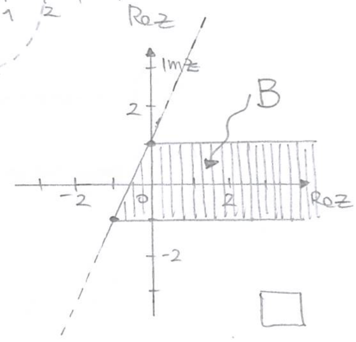
$\underline{\underline{z = \cos\dfrac{3\pi}{2} + i \sin\dfrac{3\pi}{2}}}$



1.5) i) $A = \{ z \in \mathbb{C} : |z| < 2 ,\ \operatorname{Im} z \geq 1 \}$



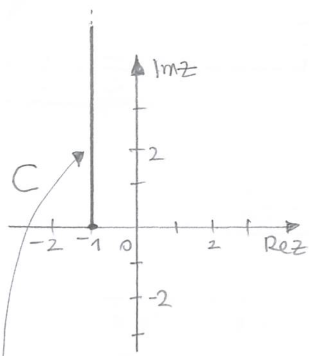
ii) $B = \{ z \in \mathbb{C} : 2\operatorname{Re} z - \operatorname{Im} z + 1 \geq 0 ,\ (\operatorname{Im} z)^2 \leq 1 \}$



$\operatorname{Re} z = x$
$\operatorname{Im} z = y$

$$\begin{cases} 2x - y + 1 \geq 0 \\ y^2 \leq 1 \end{cases} \iff \begin{cases} y \leq 2x + 1 \\ -1 \leq y \leq 1 \end{cases}$$

iii) $C = \{ z \in \mathbb{C} : |z+1| = \operatorname{Im} z \}$



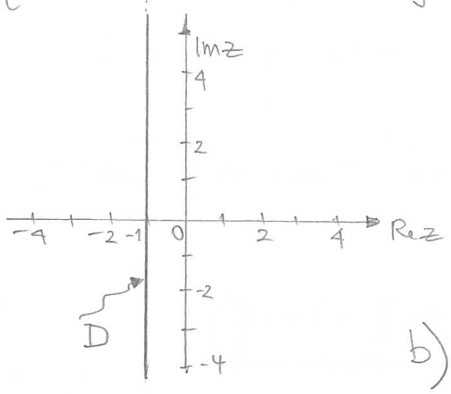
sono tutti gli $z \in \mathbb{C}$ che distano da $z_0 = -1$ esattamente $\operatorname{Im} z$.

$$|z+1| = \operatorname{Im} z \iff \begin{cases} \operatorname{Im} z \geq 0 \\ |z+1|^2 = (\operatorname{Im} z)^2 \end{cases} \iff \begin{cases} \operatorname{Im} z \geq 0 \\ |(\operatorname{Re} z + 1) + \operatorname{Im} z\, i|^2 = (\operatorname{Im} z)^2 \end{cases}$$

$$\iff \begin{cases} \operatorname{Im} z \geq 0 \\ (\operatorname{Re} z + 1)^2 + \cancel{(\operatorname{Im} z)^2} = \cancel{(\operatorname{Im} z)^2} \end{cases} \iff \begin{cases} \operatorname{Im} z \geq 0 \\ (\operatorname{Re} z + 1)^2 = 0 \end{cases}$$

$$\iff \underline{\underline{z = -1 + iy \qquad y \in \mathbb{R},\ y \geq 0}} .$$

iv) $D = \{ z \in \mathbb{C} : |z+1|^2 = (\operatorname{Im} z)^2 \} = \{ z \in \mathbb{C} : z = -1 + iy ,\ y \in \mathbb{R} \}$.



b) $C \neq D$; si ha $C \subsetneq D$.

1.6) i) $2z - 3\bar{z} = 3i + 1$: poniamo $z = x + iy$. Allora l'eq. si riscrive

$$2(x+iy) - 3(x-iy) = 3i + 1 , \text{ ossia}$$

$$2x - 3x + i(2y + 3y) = 3i + 1 , \text{ da cui segue}$$

subito che deve essere $-x = 1$ e $5y = 3$. Risulta

$\underline{\underline{z = -1 + \frac{3}{5} i}}$.

ii) $z^2 = 2\bar{z}$.

Poniamo $z = x + iy$. Allora l'equazione si riscrive $(x+iy)^2 = 2(x-iy)$, e diventa

$$x^2 + 2xyi - y^2 = 2x - 2yi,$$

ossia $(x^2 - 2x - y^2) + (2xy + 2y)i = 0 (+i0)$.

Abbiamo allora $\begin{cases} x^2 - 2x - y^2 = 0 \\ 2xy + 2y = 0 \end{cases} \iff \begin{cases} x^2 - 2x - y^2 = 0 \\ 2y(x+1) = 0. \end{cases}$

Risulta allora $y = 0$ e $x^2 - 2x = 0$

$x = -1$ e $3 - y^2 = 0$.

Le z ricercate sono allora $\underline{\underline{z = 0}}$, $\underline{\underline{z = 2}}$, $\underline{\underline{z = -1 - \sqrt{3}i}}$ e $\underline{\underline{z = -1 + \sqrt{3}i}}$. □

iii) $z^2 = 2\bar{z}i$

Poniamo $z = x + iy$. Allora l'equazione si riscrive $(x+iy)^2 = 2(x-iy)i$ e diventa

$$x^2 + 2xyi - y^2 = 2xi + 2y,$$

ossia $x^2 - y^2 - 2y + (2xy - 2x)i = 0 (+i0)$.

Risulta allora $\begin{cases} x^2 - y^2 - 2y = 0 \\ 2x(y-1) = 0 \end{cases}$.

Risulta quindi $x = 0$ e $y^2 + 2y = 0$

$y = 1$ e $x^2 - 3 = 0$.

Le z ricercate sono allora $\underline{\underline{z = 0}}$, $\underline{\underline{z = -2i}}$, $\underline{\underline{z = -\sqrt{3} + i}}$ e $\underline{\underline{z = \sqrt{3} + i}}$. ■

1.7) $z = -1 + i$

$$z = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$\underline{\underline{z^4}} = (\sqrt{2})^4\left(\cos\frac{\cancel{4}\cdot 3\pi}{\cancel{4}} + i\sin\frac{\cancel{4}\cdot 3\pi}{\cancel{4}}\right) = 4(-1) = \underline{\underline{-4}}.$$ □

$z = 1 - \sqrt{3}i$

$$z = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

$$z^4 = 2^4\left(\cos\left(-\frac{4\pi}{3}\right) + i\sin\left(-\frac{4\pi}{3}\right)\right) = 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$\underline{\underline{z^4 = -8 + 8\sqrt{3}i}}.$$ ■