Università di Trento – Dip. di Ingegneria e Scienza dell'Informazione
CdL in Informatica, Ingegneria dell'informazione e delle comunicazioni e Ingegneria dell'informazione e organizzazione d'impresa
a.a. 2017–18 – PIAZZA 7 – "altre "vecchie conoscenze": limiti – continuità – parte 1..."

1) i) $\lim_{x\to+\infty} f(x) = -1$: $\forall \varepsilon>0\ \exists K>0 : |f(x)-(-1)|<\varepsilon \quad \forall x\in \mathrm{dom} f \cap]K,+\infty[$

$\lim_{x\to -2^+} f(x) = +\infty$: $\forall M>0\ \exists \delta>0 : f(x)>M \quad \forall x \in \mathrm{dom} f, 0<x-(-2)<\delta$

$\lim_{x\to 1} f(x) = 2$: $\forall\varepsilon>0, \exists\delta>0 : |f(x)-2|<\varepsilon \quad \forall x\in\mathrm{dom} f : 0<|x-1|<\delta$. $\square$

ii) $f(x) = \frac{1}{x} - 1$ per $x\in\mathbb{R}\setminus\{0\}$

$f(x) = \frac{1}{x+2}$ per $x\in\mathbb{R}\setminus\{-2\}$

$f(x) = 2x$ per $x\in\mathbb{R}$. $\blacksquare$

2) $\lim_{x\to+\infty} \frac{2x}{x-1} = 2$. Fissiamo $\varepsilon>0$ arbitrario; dobbiamo trovare $M = M_\varepsilon > 0$: $\left|\frac{2x}{x-1} - 2\right| < \varepsilon \quad \forall x\in\mathrm{dom} f, x>M$.

Notiamo che $\left|\frac{2x}{x-1}-2\right|<\varepsilon \iff -\varepsilon<\frac{2x}{x-1}-2<\varepsilon \iff$

$-\varepsilon < \frac{2}{x-1} < \varepsilon$. Possiamo supporre che $x-1>0$ (stiamo facendo il limite per $x\to+\infty$!) e quindi $\frac{2}{x-1} > 0 > -\varepsilon$

Ora $\frac{2}{x-1}<\varepsilon \iff \frac{x-1}{2} > \frac{1}{\varepsilon} \iff x > 1+\frac{2}{\varepsilon}$. Allora basta prendere $M = 1+\frac{2}{\varepsilon}$ e si ha quanto si vuole. Per l'arbitrarietà di $\varepsilon>0$, la dim. è conclusa. $\square$

3) i) $\lim_{x\to 1} (3\overset{\nearrow 3}{x^2} - \overset{\nearrow \frac{1}{2}}{\frac{1}{x+1}}) = \underline{\frac{5}{2}}$;

$\lim_{x\to-\infty} \left(\overset{\nearrow 0}{\frac{5}{x^2+1}} + \overset{\nearrow 0}{\frac{\sin x}{x+1}}\right) = \underline{0}$;

$\lim_{x\to\frac{\pi}{2}} (\sin x + \sqrt{x^2+1}) = \sin\frac{\pi}{2} + \sqrt{\frac{\pi^2}{4}+1} = \underline{\underline{1 + \frac{\sqrt{\pi^2+4}}{2}}}$;

$\square$

ii) $\lim_{x\to-2^-} \frac{\overset{\nearrow 2}{4+x}}{\underbrace{|x|-2}_{\text{infinit. positivo}}} = \underline{\underline{+\infty}}$;

$$\lim_{x\to -2^+} \frac{\overbrace{(4+x)}^{\to 2}}{\underbrace{|x|-2}_{\text{infinitesimo negativo}}} = \underline{\underline{-\infty}}\,;$$

$$\lim_{x\to -1^-} \frac{5x^2+1}{x^3+1} = \lim_{x\to -1^-} \frac{\overbrace{5x^2+1}^{\to 6}}{(x+1)\underbrace{(x^2-x+1)}_{\to 3}} = \underline{\underline{-\infty}}\,;$$

$$\lim_{x\to 2^-} \frac{\overbrace{\log(x+1)}^{\to \log 3}}{\underbrace{x^2-4}_{\text{infinitesimo neg.}}} = \underline{\underline{-\infty}}\,.$$

□

iii) $$\lim_{x\to 0^+} \underbrace{x^2}_{\downarrow\, 0}\, \underbrace{\cos\frac{1}{x}}_{\text{limitata}} = \underline{\underline{0}}\,;$$

$$\lim_{x\to +\infty} \Big(\underbrace{x^2}_{\downarrow\, +\infty} + \underbrace{1 + \operatorname{arctg}(-x)}_{\text{limitata}}\Big) = \underline{\underline{+\infty}}\,;$$

$$\lim_{x\to -\infty} \Big(\underbrace{\sqrt[3]{|x|}}_{\downarrow\, +\infty} + \underbrace{\sin x\cos x}_{\text{limitata}}\Big) = \underline{\underline{+\infty}}\,.$$

■

4) $$\lim_{x\to -\infty} \frac{\overbrace{\sin x\cos x}^{\text{limitata}}}{\underbrace{x^2}_{\to +\infty}} = \underline{\underline{0}}\,; \qquad \lim_{x\to +\infty} \frac{\overbrace{\operatorname{arctg} x^3}^{\text{limitata}}}{\underbrace{\sqrt{x}+1}_{\to +\infty}} = \underline{\underline{0}}\,;$$

$$\lim_{x\to -\infty} \Big(\underbrace{x}_{\downarrow\, -\infty} - \underbrace{\operatorname{arctg} x}_{\text{limitata}}\Big) = \underline{\underline{-\infty}}\,; \qquad \lim_{x\to 0^-} \underbrace{\sqrt[3]{x}}_{\downarrow\, 0}\, \underbrace{\cos\frac{1}{x^2}}_{\text{limitata}} = \underline{\underline{0}}\,.$$

■

5) i) $$\lim_{x\to 0} \frac{x^3+2x^2}{4x^4+x^5} \underset{\left[\frac{0}{0}\right]}{=} \underline{\underline{+\infty}}$$: infatti $$\frac{\cancel{x^2}\overbrace{(2+x)}^{\to 2}}{x^{\cancel{4}2}\underbrace{(4+x)}_{\to 4}} \xrightarrow[x\to 0]{} +\infty\,;$$

$$\lim_{x\to -2^-} \left(\frac{1}{x+2} + \frac{1}{x^2-4}\right) \underset{[-\infty+\infty]}{=} \underline{\underline{-\infty}}$$: infatti
$$\frac{1}{x+2} + \frac{1}{x^2-4} = \frac{x-2+1}{x^2-4} = \frac{\overbrace{(x-1)}^{\to -3}}{\underbrace{(x^2-4)}_{\text{infinitesimo positivo}}} \xrightarrow[x\to -2^-]{} -\infty\,;$$

□

ii) $$\lim_{x\to +\infty} \frac{x^4+3x}{x-1-2x^4} \underset{\left[\frac{\infty}{\infty}\right]}{=} \underline{\underline{-\frac{1}{2}}}$$: $$\frac{\cancel{x^4}\left(1+\frac{3}{x^3}\right)}{\cancel{x^4}\left(\frac{1}{x^3}-\frac{1}{x^4}-2\right)} \xrightarrow[x\to +\infty]{} -\frac{1}{2}\,;$$

$$\lim_{x\to 0^+} \frac{x+3\sqrt{x}}{2x-6\sqrt{x}} \underset{\left[\frac{0}{0}\right]}{=} \underline{\underline{-\frac{1}{2}}}$$: infatti $$\frac{\cancel{\sqrt{x}}(\sqrt{x}+3)}{\cancel{\sqrt{x}}(2\sqrt{x}-6)} \longrightarrow -\frac{3}{6}\,;$$

$$\lim_{x\to -\infty} \frac{x+3\sqrt{|x|}}{2x-6\sqrt{|x|}} \underset{\left[\frac{\infty}{\infty}\right]}{=} \underline{\underline{\frac{1}{2}}}$$: infatti $$\frac{\cancel{x}\left(1+3\cdot\frac{1}{\sqrt{|x|}}\right)}{\cancel{x}\left(2-6\cdot\frac{1}{\sqrt{|x|}}\right)} \xrightarrow[x\to -\infty]{} \frac{1}{2}\,;$$

□

iii) $$\lim_{x\to +\infty} \frac{x^4-x}{\sqrt{x}+3x^4} \underset{\left[\frac{\infty}{\infty}\right]}{=} \underline{\underline{\frac{1}{3}}}$$

$$\lim_{x\to -\infty} \left(\frac{x^2}{x-1} - \frac{x^3}{x^2-1}\right) \underset{[-\infty+\infty]}{=} \underline{\underline{1}}$$ infatti $$\frac{x^2}{x-1} - \frac{x^3}{x^2-1} = \frac{x^2(x+1)-x^3}{x^2-1} = \frac{x^2}{x^2-1} \xrightarrow[x\to -\infty]{} 1\,;$$

□

iv) $\lim\limits_{x\to-\infty} \dfrac{\operatorname{arctg} x - x^2}{2+x^k} \underset{\left[\frac{\infty}{\infty}\right]}{=} \begin{cases} +\infty & k=1 \\ -1 & k=2 \\ 0 & k=3 \end{cases}$

$\lim\limits_{x\to-1} \dfrac{1+x}{\sqrt{-x}-1} \underset{\left[\frac{0}{0}\right]}{=} \underline{\underline{-2}}$. $\quad : \quad \dfrac{(1+\sqrt{-x})\cancel{(1-\sqrt{-x})}}{-\cancel{(1-\sqrt{-x})}} \xrightarrow[x\to-1]{} -2.$

$\lim\limits_{x\to0^+} \dfrac{\sqrt{x}}{\sqrt{x+2}-\sqrt{2}} \underset{\left[\frac{0}{0}\right]}{=} \underline{\underline{+\infty}} \quad : \quad \dfrac{\sqrt{x}\,(\sqrt{x+2}+\sqrt{2})}{x} = \dfrac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x}} \xrightarrow[x\to0^+]{} +\infty$

v) $\lim\limits_{x\to1} \dfrac{\sqrt{2+x}-\sqrt{4-x}}{x-1} \underset{\left[\frac{0}{0}\right]}{=} \underline{\underline{\frac{1}{\sqrt{3}}}} \quad : \quad \dfrac{\sqrt{2+x}-\sqrt{4-x}}{(x-1)} \cdot \dfrac{\sqrt{2+x}+\sqrt{4-x}}{\sqrt{2+x}+\sqrt{4-x}} =$

$= \dfrac{2+x-4+x}{(x-1)(\sqrt{2+x}+\sqrt{4-x})} = \dfrac{2\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{2+x}+\sqrt{4-x})}$

$\xrightarrow[x\to1]{} \dfrac{\cancel{2}}{\cancel{2}\sqrt{3}} = \dfrac{1}{\sqrt{3}}$;

$\lim\limits_{x\to+\infty} \dfrac{1}{x^\alpha(\sqrt{x+2}-\sqrt{x-2})} = \dfrac{\sqrt{x+2}+\sqrt{x-2}}{x^\alpha(\sqrt{x+2}-\sqrt{x-2})} \cdot \dfrac{1}{\sqrt{x+2}+\sqrt{x-2}} =$

$= \begin{cases} 0 & \text{se } \alpha > \frac{1}{2} \\ \frac{1}{2} & \text{se } \alpha = \frac{1}{2} \\ +\infty & \text{se } \alpha < \frac{1}{2} \end{cases} \quad = \dfrac{\sqrt{x+2}+\sqrt{x-2}}{x^\alpha \cdot 4} = \dfrac{x^{1/2}\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}\right)}{4x^\alpha}$ ■

6) i) $A = \left\{x_n = \cos\left(\frac{2}{n+3}\right) : n\in\mathbb{N}, n\geq 0\right\}$: $\quad |x_n| \leq 1 \quad \forall n\in\mathbb{N}, n\geq 0$
quindi A è un insieme limitato.

Inoltre $\frac{2}{n+3} \searrow$, $\cos x \searrow$ su $[0,\frac{\pi}{2}] \Rightarrow x_n \nearrow$.

Quindi, dal teorema di Esistenza del limite per funz. monotone segue $\underline{\underline{\inf A}} = \underline{\underline{\min A}} = x_0 = \underline{\underline{\cos\frac{2}{3}}}$

$\underline{\underline{\sup A}} = \lim\limits_{n\to+\infty} \cos\left(\frac{2}{n+3}\right) = \cos 0 = \underline{\underline{1}}$. □

ii) $B = \left\{x_n = \operatorname{arctg}\left(\frac{n-2}{n}\right) : n\in\mathbb{N}, n\geq 1\right\}$ $\quad |x_n| \leq \frac{\pi}{2} \quad \forall n\in\mathbb{N}, n\geq 1$
quindi B è un insieme limitato.

Inoltre $\frac{n-2}{n} = 1-\frac{2}{n} \nearrow$, $\operatorname{arctg} x \nearrow \Rightarrow x_n \nearrow$.

Come sopra si ha

$\underline{\underline{\inf B}} = \underline{\underline{\min B}} = x_1 = \operatorname{arctg}(-1) = \underline{\underline{-\frac{\pi}{4}}}$

$\underline{\underline{\sup B}} = \lim\limits_{n\to+\infty} x_n = \operatorname{arctg} 1 = \underline{\underline{\frac{\pi}{4}}}$. □

iii) $C = \{ x_n = e^{\arcsin(1-\frac{1}{n})} : n \in \mathbb{N}, n \geq 1 \}$

Ora $|\arcsin x| \leq \frac{\pi}{2}$ $\forall x \in [-1,1]$ e quindi $|x_n| \leq e^{\pi/2}$ $\forall n \in \mathbb{N}, n \geq 1$

Quindi C è un insieme limitato.

Ora $1-\frac{1}{n} \nearrow$ $\arcsin x \nearrow$ $e^x \nearrow$ $\Rightarrow$ x_n è crescente.

Risulta allora come sopra

$$\underline{\underline{\inf C}} = \min C = x_1 = e^{\arcsin 0} = \underline{\underline{1}}$$

$$\underline{\underline{\sup C}} = \lim_{n\to+\infty} e^{\arcsin(1-1/n)} = e^{\arcsin 1} = \underline{\underline{e^{\pi/2}}}.$$ □

iv) $D = \{ x_n = \log_{1/2} \arcsin(1-\frac{1}{n}) : n \in \mathbb{N}, n \geq 2 \}$

Ora $\frac{1}{2} < 1 - \frac{1}{n} < 1$ $\forall n \geq 2$ $\underset{\arcsin x \nearrow}{\Rightarrow}$ $\arcsin\frac{1}{2} < \arcsin(1-\frac{1}{n}) < \arcsin 1$ $\nearrow^{0}$

$\underset{\log_{1/2} x \searrow}{\Rightarrow}$ $\log_{1/2}(\overbrace{\arcsin \tfrac{1}{2}}^{\pi/6}) > x_n > \log_{1/2}(\underbrace{\arcsin 1}_{\pi/2})$ $\forall n \in \mathbb{N}, n \geq 2$

Quindi D è limitato.

Oss. che $\log_{1/2} x \searrow$, in questo caso, risulta $x_n \searrow$ e quindi

$$\underline{\underline{\sup D}} = \max D = x_2 = \underline{\underline{\log_{1/2}\left(\tfrac{\pi}{6}\right)}}.$$

$$\underline{\underline{\inf D}} = \lim_{n\to+\infty} x_n = \log_{1/2}(\arcsin 1) = \underline{\underline{\log_{1/2}\left(\tfrac{\pi}{2}\right)}}.$$ ■

7) $f(x) = \begin{cases} \sin x - \alpha & \text{se } x<0 \\ -2 & x=0 \\ x^2-2 & \text{se } x>0 \end{cases}$ $\lim_{x\to0^-}(\sin x - \alpha) = -\alpha = -2 = \lim_{x\to0^+}(x^2-2)$

$\Leftrightarrow$ $\underline{\underline{\alpha = 2}}$

Basta controllare la continuità in $x=0$, in tutti gli altri pt. f è costituito da funz. elem. continue. □

$f(x) = \begin{cases} \alpha(\sin x - 1) & \text{se } x\leq 0 \\ x^2-3x+2 & \text{se } x>0 \end{cases}$ $\lim_{x\to0^-} \alpha(\sin x - 1) = -\alpha = f(0) = \lim_{x\to0^+}(x^2-3x+2) = 2$

$\Leftrightarrow$ $\underline{\underline{\alpha = -2}}$. ■

8) Basta controllare la continuità nei pt. dove si devono attaccare!

$$\begin{cases} \lim_{x\to0^-} \frac{2}{\pi}\operatorname{arctg}\frac{1}{x} = \alpha\cos 0 + \beta \sin 0^{(1)} \\ \lim_{x\to\frac{\pi}{2}^-}(\alpha\cos x + \beta\sin x) = \cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right)^{(2)} \end{cases} \Leftrightarrow \begin{cases} -1 = \alpha \\ \beta = -1 \end{cases} \Rightarrow \underline{\underline{\alpha = \beta = -1}}$$ □

$$\begin{aligned} &\lim_{x\to0^-}(x^2+1) = 2\cdot 0 + \beta^{(3)} \\ &\lim_{x\to1^+}\log(x+|\alpha|) = 2+\beta^{(4)} \end{aligned} \Leftrightarrow \begin{cases} \underline{1 = \beta} \\ \log(1+|\alpha|) = 3 \end{cases} \Leftrightarrow 1+|\alpha| = e^3 \quad \underline{\underline{\alpha = \pm(e^3-1)}}.$$ ■

(1) $\alpha\cos x + \beta\sin x$ è continua da destra in $x=0$.
(2) $\cos(\pi/2 + x)$ è continua da destra in $x=\pi/2$.
(3) $2x+\beta$ è continua da destra in $x=0$
(4) $2x+\beta$ è continua da sinistra in $x=1$.