Es. 11 - 10/12/19

Esercizi su integrali

1. Funzione integrale

a) Studiate il limite di

$$\boxed{\lim_{x\to 0} \frac{\int_0^x \left[\cos t - e^{t^2/2} + \log^2(1+t)\right] dt}{x^4}}$$

SOL Taylor

|| $\cos t = \cancel{1} - \frac{t^2}{2} + \frac{t^4}{24} + o(t^4)$

|| $e^{t^2/2} = \cancel{1} + \frac{t^2}{2} + \frac{t^4}{8} + o(t^4)$

|| $\log^2(1+t) = \left[t - \frac{t^2}{2} + o(t^2)\right]^2 = t^2 - t^3 + \frac{t^4}{4} + o(t^4)$

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + \dots$

$[\dots] = \cancel{1} - \cancel{1} - \cancel{\frac{t^2}{2}} - \cancel{\frac{t^2}{2}} + \frac{t^4}{24} - \frac{t^4}{8} + \cancel{t^2} - t^3 + \frac{t^4}{4} + \dots$

$\Rightarrow \lim_{x\to 0} \frac{\int_0^x [\dots]\, dt}{x^4} = \lim_{x\to 0} \frac{\int_0^x -t^3}{x^4} + \underbrace{\text{Resto}}_{\to 0}$

$= \lim_{x\to 0} \frac{-\cancel{x^4}}{4 \cdot \cancel{x^4}}$ $\qquad \int f' = f$

$= -\frac{1}{4}$ //

Altro modo: con L'Hopital

$F(x) = \int_0^x [\dots]\,dt$ è cont. & derivabile

$G(x) = x^4$ — " —

$\lim_{x\to 0} \frac{F(x)}{G(x)}$ $\left[\frac{0}{0}\right]$ F.I.

$$\lim_{x\to 0} \frac{F'(x)}{G'(x)} = \lim_{x\to 0} \frac{\cos x - e^{x^2/2} + \log^2(1+x)}{4x^3}$$

come sopra con Taylor ...

$\blacksquare$

(b) Determinare i pti. critici e la concavità e convessità di

$$\boxed{F(x) = \int_0^{x} \sqrt[3]{t}\cdot e^{-t}\,dt \quad , \quad x \in [1/4, 3]}$$

<u>Sol</u> $t \to \sqrt[3]{t}\cdot e^{-t}$ cont. e derivabile $(t \neq 0)$

$\Rightarrow F$ è cont. e derivabile (là dove serve ☺)

pti. critici $\rightsquigarrow F'$

conv/conc $\rightsquigarrow F''$ studio del segno

$$F'(x) = \underbrace{\sqrt[3]{x}}_{>0}\cdot \underbrace{e^{-x}}_{\neq 0,\ >0} \qquad x \in [1/4, 3]$$

$F(x) = \int_{x_0}^{x} f(t)\,dt$

$F'(x) = f(x)$

$\Rightarrow \nexists$ pti. critici

$$F''(x) = \frac{1}{3}x^{-2/3}\cdot e^{-x} - x^{\frac{1}{3}}\cdot e^{-x}$$

$$= \underbrace{e^{-x}}_{>0}\cdot \underbrace{\left[\frac{1}{3}x^{-2/3} - x^{1/3}\right]}$$

$(x^p)' = p\cdot x^{p-1}$

$(e^{-x})' = e^{-x}\cdot(-1)$

Studio del segno

$$\frac{1}{3}\cdot\frac{1}{\sqrt[3]{x^2}} - \sqrt[3]{x} = \frac{1}{3}\,\frac{1 - 3\cdot x}{\underbrace{\sqrt[3]{x^2}}_{>0}}$$

+ 0 −

1/4 1/3 3

$\Rightarrow$ F è convessa su $[1/4, 1/3]$

F è concava su $[1/3, 3]$.

∎

Es. 2) Integrali definiti

a) $\boxed{\int_0^1 \left(3\sin x - \frac{3}{1+x^2}\right)dx}$

$(-\cos x)' = \sin x$

$(\arctan x)' = \frac{1}{1+x^2}$

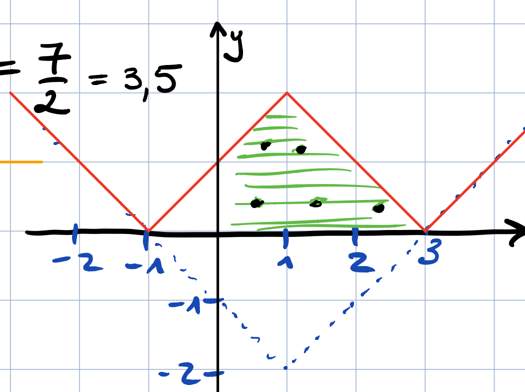
$$= 3\cdot(-\cos x]_0^1 - 3\cdot(\arctan x]_0^1$$

$(f(x)|_a^b = f(b) - f(a)$

$$= 3\cdot(1 - \cos(1)) - 3\cdot\arctan 1$$

∎

b) $\boxed{\int_0^3 ||x-1|-2|\,dx = \frac{7}{2} = 3{,}5}$



∎

(c) $\boxed{\frac{1}{2}\int_1^2 \underbrace{\frac{2x}{1+x^2}}_{f(x)}\,dx} = I$

SOL $f(x) = \left[\frac{1}{2}\log(1+x^2)\right]'$

$$\Rightarrow I = \frac{1}{2}\log(1+2^2) - \frac{1}{2}\log(1+1^2)$$
$$= \frac{1}{2}\log(5) - \frac{1}{2}\log(2).$$

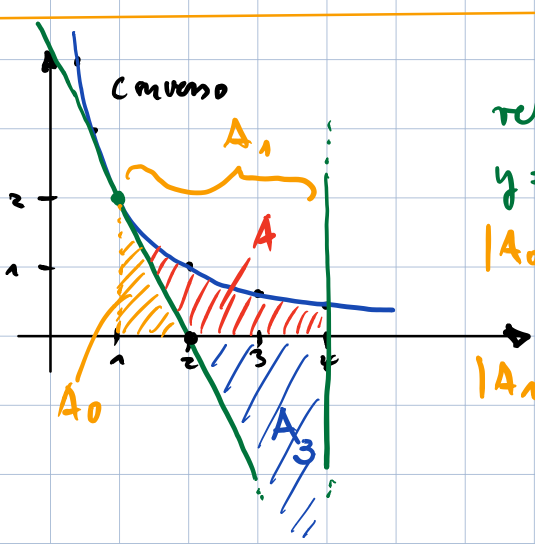
(d) $\boxed{\frac{1}{3}\int_0^\pi 3\cdot \underbrace{2x\cdot\cos(3x^2-1)}_{f(x)}\,dx} = I$

SOL: $f(x) = \left[\frac{\sin(3x^2-1)}{3}\right]'$

$$\Rightarrow I = \left(\frac{1}{3}\sin(3x^2-1)\right|_0^\pi \ldots$$

(e) Calcolate l'area delimitata dal grafico di $f(x) = \frac{2}{x}$ e della retta tang. al grafico di f in $(1,2)$ e della retta $x = 4$

SOL.



retta tangente

$y = -2x+4$ (verifica!)

$|A_0| = 1$

$$|A_1| = \int_1^4 \frac{2\,dx}{x} = 2\cdot\log x\Big|_1^4 = 2\cdot\log(4)$$

$\Rightarrow |A| = |A_1| - |A_0| = 2\cdot\log(4) - 1$

$\text{area}(A \cup A_3) = |A| + |A_3| = |A| + \frac{\cancel{2}\cdot 4}{\cancel{2}} = \underline{2\log 4 + 3}.$ ■

Es. 3) <u>Integrazione per parti</u> (Int. indef.)

a) $\boxed{\int \log^2 x \, dx} = I$
$\cdot (x)'$

<u>Sol.</u>

$\int f' \cdot g = f\cdot g - \int f \cdot g'$

$I = x \cdot \log^2 x - \int \cancel{x} \cdot 2 \cdot \log x \cdot \frac{1}{\cancel{x}} \, dx$

$= x \cdot \log^2 x - 2 \int \log x \, dx$
$\cdot (x)'$

$= —''— - 2 \cdot \left[\log x \cdot x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx \right]$

$= x \cdot \log^2 x - 2x \cdot \log x + 2x + c$ ■

(b) $\boxed{\underbrace{\int e^x \cdot \cos(2x) \, dx}_{I}}$ $= e^x \cdot \cos(2x) - 2\int e^x \cdot (-\sin 2x) \, dx$
$(e^x)'$

$= e^x \cdot \cos(2x) + 2 \underline{\int e^x \cdot \sin 2x \, dx}$

$\int e^x \cdot \sin 2x \, dx = \underline{e^x \cdot \sin(2x) - 2\underbrace{\int e^x \cdot \cos(2x) \, dx}_{I}}$
$(e^x)'$

$\Rightarrow I = e^x \cdot \cos(2x) + 2 \cdot e^x \cdot \sin(2x) - 4 \cdot I$

$\Rightarrow \therefore \cancel{5} \cdot I = \frac{e^x \cdot \cos(2x) + 2 \cdot e^x \sin(2x)}{5}$ $|:5$ ■

(c) $\boxed{\int x \cdot \arctan x \, dx}$

SOL: $F(x) = \frac{x^2}{2} \cdot \arctan x - \frac{x}{2} + \frac{\arctan x}{2} + c$

HINT: $x = \left(\frac{x^2}{2}\right)'$ ...

Es. 4) Integr. per sostituzione

(a) $\boxed{\int \frac{x}{\sqrt{1+x}} \, dx = F(x)}$

SOL: sia $t = 1+x$ che $t = \sqrt{1+x}$ vanno bene

$\Rightarrow t^2 = 1+x \Leftrightarrow t^2 - 1 = x \quad (x(t), \varphi(t))$

$dx = 2t \cdot dt$

$$F(x) = \int \frac{t^2-1}{\cancel{t}} \cdot 2\cancel{t} \cdot dt = 2 \cdot \int t^2 - 1 \, dt$$

$$= 2 \cdot \left(\frac{t^3}{3} - t \right) \underset{\text{Resostituz.}}{=} 2 \cdot \left(\frac{(1+x)^{3/2}}{3} - \sqrt{1+x} \right) + c$$

(b) $\boxed{\int e^{2x} \cdot \log(1+e^x) \, dx} = F(x)$

SOL: sia $t = e^x$ che $t = 1+e^x$ vanno bene

$\Rightarrow \log t = x \quad ; \quad dx = \frac{1}{t} \cdot dt$

$$\Rightarrow F(x(t)) = \int \underbrace{t^{\cancel{3}}}_{\left(\frac{t^2}{2}\right)'} \cdot \log(1+t) \cdot \frac{1}{\cancel{t}} \cdot dt$$

$$= \frac{t^2}{2} \cdot \log(1+t) - \int \frac{t^2}{2} \cdot \frac{1}{1+t}\, dt$$

I.P.P.

$$\overset{(t-1)\cdot\cancel{(t+1)}}{=} \quad \text{—— '' ——} \quad - \frac{1}{2} \int \frac{t^2-1}{\cancel{1+t}} + \frac{1}{1+t}\, dt$$

$$= \quad \text{—— '' ——} \quad - \frac{1}{2}\left[\frac{t^2}{2} - t + \log(1+t)\right] + c$$

Fai tu la risostituzione!

$\blacksquare$

(c) Sostituzione speciale con fz. trigonometr.

$$\boxed{\int \frac{\tan x}{\sin x + \tan x}\, dx} \underset{?!}{\sim} \int \frac{dx}{1+\cos x} = F(x)$$

Sol. $t = \tan\left(\frac{x}{2}\right)$

$$\cos x = \frac{1-t^2}{1+t^2} \quad ; \quad x = 2 \cdot \arctan t$$

$$dx = \frac{2}{1+t^2} \cdot dt$$

$$\Rightarrow F(x(t)) = \int \frac{2/(1+t^2)}{1 + \frac{1-t^2}{1+t^2}}\, dt$$

$$= \int \frac{\cancel{2} \cdot \cancel{(1+t^2)}}{\cancel{2}} \cdot \frac{1}{\cancel{1+t^2}}\, dt$$

$$= t + c \underset{\text{R.Sost.}}{=} \tan(x/2) + c$$

$\blacksquare$

$$\boxed{\int \frac{\sin x}{3+\sin x}\,dx} \quad , \quad t=\tan\left(\frac{x}{2}\right)$$

$$= \int \frac{\sin x+3-3}{3+\sin x}\,dx = \int 1\,dx - 3\int \frac{1}{3+\sin x}\,dx$$

$$\int \frac{1}{3+\sin x}\,dx = \int \frac{1}{3+\frac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt = \int \frac{2}{3+3t^2+2t}\,dt$$

$t=\mathrm{tg}\frac{x}{2}$

$x=2\,\mathrm{arctg}\,t$

$dx=\frac{2}{1+t^2}\,dt$

$\sin x=\frac{2t}{1+t^2}$

$$= \frac{2}{3}\int \frac{1}{t^2+\frac{2}{3}t+1}\,dt$$

$$= \frac{2}{3}\int \frac{1}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}}\,dt =$$

$$= \frac{2}{3}\cdot\frac{1}{8/9}\int \frac{1}{\left(\frac{t+1/3}{\sqrt{8}/3}\right)^2+1}\,dt$$

$$= \frac{2}{3}\,\frac{1}{\sqrt{8}/3}\int \frac{\frac{\sqrt{8}}{3}}{\left(\frac{t+1/3}{\sqrt{8}/3}\right)^2+1}\,dt$$

$$= \frac{2}{\sqrt{8}}\,\mathrm{arctg}\left(\frac{t+1/3}{\sqrt{8}/3}\right)+c\,.$$

$$\Rightarrow \int \frac{\sin x}{3+\sin x}\,dx = x-3\cdot\frac{2}{\sqrt{8}}\,\mathrm{arctg}\left(\frac{\mathrm{tg}\frac{x}{2}+\frac{1}{3}}{\sqrt{8}/3}\right)+c$$