STUDIO DI FUNZIONE E GRAFICI QUALITATIVI

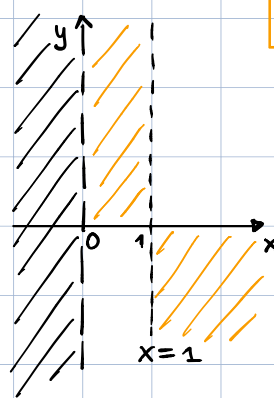
$$\boxed{f(x)=\frac{x}{\log^3|x|}}$$

- $\mathrm{dom} f=\{x\in\mathbb{R}: x\neq 0,\ x\neq \pm 1\}$
- f dispari $\quad f(-x)=-f(x)\ \ \forall x\in \mathrm{dom} f$

basta studiare $f(x)$ per $x>0$

Studiamo $f\big|_{\mathrm{dom} f\,\cap\,]0,+\infty[}$ $=]0,1[\,\cup\,]1,+\infty[$

$$\boxed{g(x)=\frac{x}{\log^3 x}} \quad \text{su }]0,1[\,\cup\,]1,+\infty[$$



- segno g

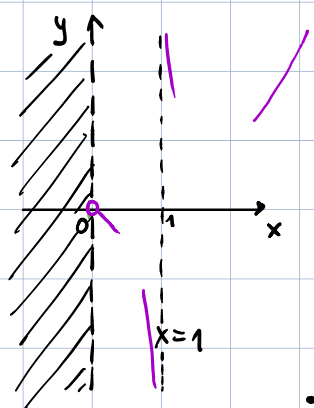
$$/\!/\!/\ \underset{0}{\circ}\ -\ \underset{1}{\circ}\ +\ +\ +\ \longrightarrow \quad g$$

- $\lim\limits_{x\to 0^+} g(x)=0$

$$\left.\begin{array}{l}\lim\limits_{x\to 1^-} g(x)=-\infty \\[2em] \lim\limits_{x\to 1^+} g(x)=+\infty\end{array}\right\}$$

$x=1$ asint. verticale per f (da destra/sinistra)

$$\lim_{x\to+\infty} g(x)=+\infty$$



- $\mathrm{dom}\, g'=]0,1[\,\cup\,]1,+\infty[$

$$g'(x)=\frac{\log^3 x-\cancel{x}(3\log^2 x)\frac{1}{\cancel{x}}}{\log^6 x}=\frac{\log x-3}{\log^4 x}$$

$g'(x)=0 \iff \log x=3 \iff x=e^3$ $\quad /\!/\ \underset{0}{\circ}\ -\ \underset{1}{\circ}\ -\ \underset{e^3}{\bullet}\ +\ +\ \to g'$

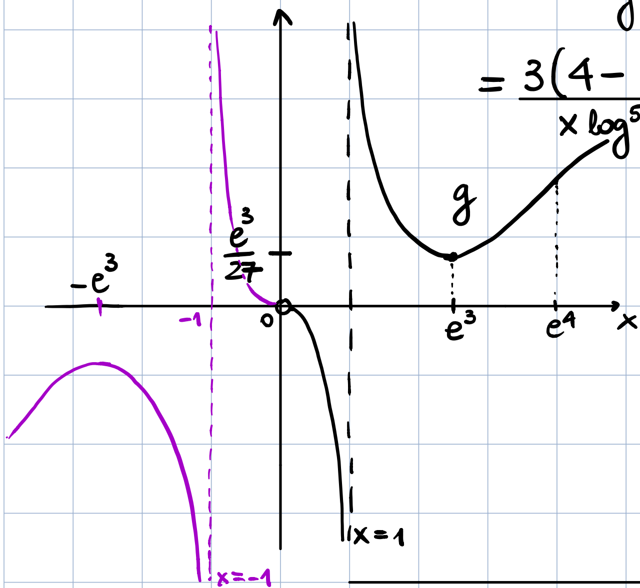
$\searrow \quad \searrow \quad \nearrow \quad g$

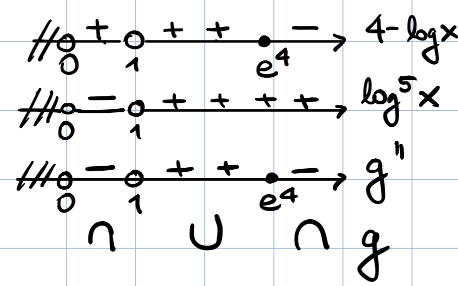
$g(e^3)=\frac{e^3}{27}<1$ $\quad$ pt. di min. loc. stretto per g

- $\text{dom}\, g'' =]0,1[\cup]1,+\infty[$

$$g''(x) = \frac{\frac{1}{x}\cdot \log^{\cancel{4}} x - (\log x - 3)\, 4\cancel{\log^3 x}\,\frac{1}{x}}{\log^{\cancel{8}\,5} x} = \frac{\log x - 4\log x + 12}{x \log^5 x}$$

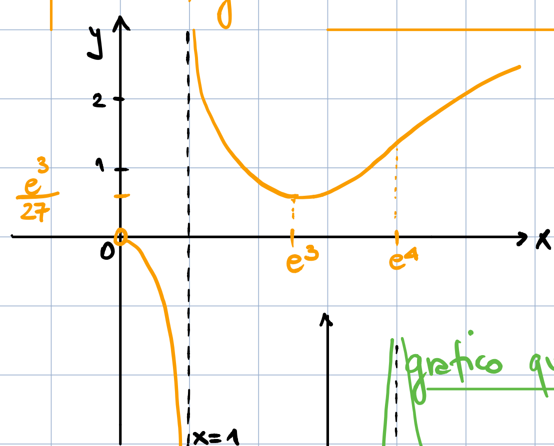
$$= \frac{3(4-\log x)}{x\log^5 x}$$





Considenamo solo

$$\boxed{g(x) = \frac{x}{\log^3 x} \quad \text{su }]0,1[\cup]1,+\infty[}$$



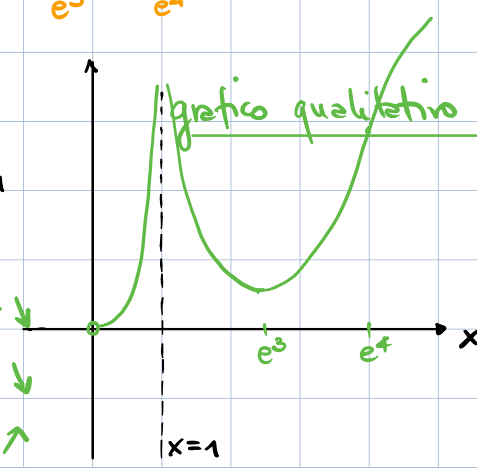
$\boxed{(g^2)' = (2g)g'}$

$g>0 \quad g\searrow \;\Rightarrow\; g^2 \searrow$

$g<0 \quad g\nearrow \;\Rightarrow\; g^2 \searrow$

$g<0 \quad g\searrow \;\Rightarrow\; g^2 \nearrow$

grafico qualitativo di $g^2(x)$

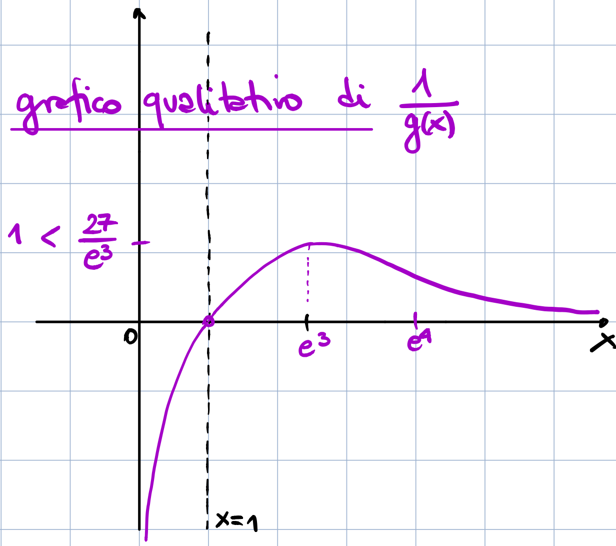


$g<0 \;\Rightarrow\; g^2>0$

$g\to\pm\infty \;\Rightarrow\; g^2\to+\infty$

$g\to 0 \;\Rightarrow\; g^2\to 0$

grafico qualitativo di $\frac{1}{g(x)}$



$g \to 0^{\pm} \Rightarrow \frac{1}{g} \to \pm\infty$

$g \to \pm\infty \Rightarrow \frac{1}{g} \to 0^{\pm}$

$g = 1 \Rightarrow \frac{1}{g} = 1$

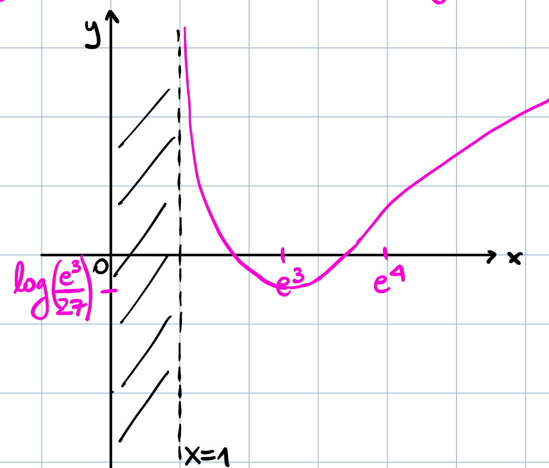
$g \nearrow \Rightarrow \frac{1}{g} \searrow$ e viceversa

$$\boxed{\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g^2(x)}}$$

$g > 0 \Rightarrow \frac{1}{g} > 0$

$g < 0 \Rightarrow \frac{1}{g} < 0$

grafico qualitativo di $\log(g(x))$



$$\boxed{(\log g(x))' = \underbrace{\frac{1}{g(x)}}_{>0} g'(x)}$$

$\exists$ solo per $g(x) > 0$

log $\nearrow$ quindi conserva la monotonia di $g(x)$

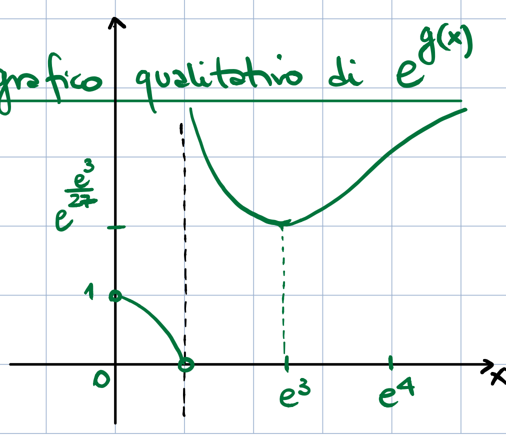
$g < 1 \Rightarrow \log(g) < 0$

$g > 1 \Rightarrow \log(g) > 0$

$g \to 0^+ \Rightarrow \log(g) \to -\infty$

$g \to +\infty \Rightarrow \log(g) \to +\infty$

grafico qualitativo di $e^{g(x)}$



Sempre > 0 ; $g = 0 \Rightarrow e^g = 1$

$g \to +\infty \Rightarrow e^g \to +\infty$

$g \to -\infty \Rightarrow e^g \to 0$

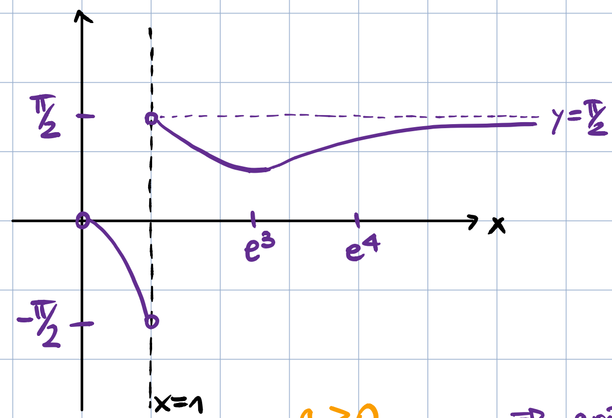
$g \nearrow \Rightarrow e^g \nearrow$

$g \searrow \Rightarrow e^g \searrow$

$$\boxed{\left(e^{g(x)}\right)' = e^{g(x)} g'(x)}$$

$e^x \nearrow$ mantiene la monot.

grafico qualitativo di arctg g(x)



$$\boxed{(\operatorname{arctg} g)' = \frac{1}{1+g^2}\, g'}$$

arctg $\nearrow$, quindi mantiene la monotonia di g

$g>0 \Rightarrow \operatorname{arctg} g > 0$

$g<0 \Rightarrow \operatorname{arctg} g < 0$

$g=0 \Rightarrow \operatorname{arctg} g = 0$

$g \to \pm\infty \Rightarrow \operatorname{arctg} g \to \pm\frac{\pi}{2}$ ■

LIMITI

Es. 1 Per quali $\alpha>0$ $\exists$ finito $\boxed{\lim_{x\to0^+} \frac{x^3+\log^2(1+x)-x^2}{x^\alpha}}$ $\left[\frac{0}{0}\right]$

$$\frac{x^3+\log^2(1+x)-x^2}{x^\alpha} = \frac{x^3+\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\right)^2-x^2}{x^\alpha}$$

$\log(1+t)=t-\frac{t^2}{2}+\frac{t^3}{3}+o(t^3) \quad t\to0$

$$= \frac{\cancel{x^3}+\cancel{x^2}+\frac{x^4}{4}-\cancel{x^3}+\frac{2}{3}x^4+o(x^4)-\cancel{x^2}}{x^\alpha}$$

$$= \frac{\left(\frac{1}{4}+\frac{2}{3}\right)x^4+o(x^4)}{x^\alpha}$$

$\Rightarrow \lim_{x\to0^+} \frac{x^3+\log^2(1+x)-x^2}{x^\alpha}$ $\exists$ finito se $0<\alpha\le4$ e vale $\begin{cases} 0 & \text{se } \alpha<4 \\ \frac{11}{12} & \text{se } \alpha=4 \end{cases}$ □

Es. i) $\lim_{x\to 0^+} \dfrac{\int_0^x (e^t-1)\,dt}{x^2} = \left[\frac{0}{0}\right]$

$$\frac{f'(x)}{g'(x)} = \frac{e^x-1}{2x} \xrightarrow[x\to 0^+]{} \frac{1}{2} \overset{H}{\Longrightarrow} \lim_{x\to 0^+} \frac{\int_0^x (e^t-1)\,dt}{x^2} = \frac{1}{2}. \quad \square$$

ii) $\lim_{x\to 0^+} \dfrac{\int_x^{x^2} (e^t-1)\,dt}{x^2} = \left[\frac{0}{0}\right]$

$$\frac{f'(x)}{g'(x)} = \frac{(e^{x^2}-1)\,2x - (e^x-1)\cdot 1}{2x} = e^{x^2}-1 - \frac{e^x-1}{2x} \xrightarrow[x\to 0]{} -\frac{1}{2}$$

$$\overset{H}{\Longrightarrow} \lim_{x\to 0^+} \frac{\int_x^{x^2} (e^t-1)\,dt}{x^2} = -\frac{1}{2} \quad \blacksquare$$

POLINOMIO DI TAYLOR (ordine n, centrato in x_0) $\qquad \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = P_n(x)$

SVILUPPO DI TAYLOR $\qquad P_n(x) + o((x-x_0)^n)$

Es i) Polinomio di Taylor, ordine 3, in $x_0=0$

$$f(x) = \frac{3}{2}x - 1 + \cos\frac{x}{2} - \log\left(1+\frac{x}{2}\right)$$

ii) Polinomio di Taylor, ordine 2, in $x_0=\pi$, $\quad g(x) = \frac{3}{2}x + \cos\frac{x}{2}$.

i) $P_3(x) = \frac{3}{2}x - 1 + \left(1 - \frac{x^2}{4\cdot 2}\right) - \left(\frac{x}{2} - \frac{x^2}{2\cdot 4} + \frac{x^3}{3\cdot 8}\right)$ $\quad$ [uso quelli noti]

$$= \frac{3}{2}x - \cancel{1} + \cancel{1} - \cancel{\frac{x^2}{8}} - \frac{x}{2} + \cancel{\frac{x^2}{8}} - \frac{x^3}{24} = \underline{x - \frac{x^3}{24}}$$

ii) $P_2(x) = \sum_{k=0}^{2} \frac{g^{(k)}(\pi)}{k!}(x-\pi)$ $\qquad g'(x) = \frac{3}{2} - \frac{1}{2}\sin\frac{x}{2}$

$g''(x) = -\frac{1}{4}\cos\frac{x}{2}$

$$P_2(x) = g(\pi) + g'(\pi)(x-\pi) + g''(\pi)\frac{(x-\pi)^2}{2}$$
$$= \frac{3}{2}\pi + \left(\frac{3}{2}-\frac{1}{2}\right)(x-\pi)$$
$$= \underline{\frac{3}{2}\pi + (x-\pi)} \, .$$

■

SERIE

Es. Det. $\alpha \in \mathbb{R}$ il carattere della serie $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^\alpha}\underbrace{\left(1-\cos\frac{1}{n}\right)}_{a_n > 0}}$

$$a_n \sim \frac{1}{n^\alpha} \cdot \frac{1}{2n^2} = \frac{1}{2n^{2+\alpha}} \qquad n \to +\infty$$

la serie $\sum_{n=1}^{\infty} \frac{1}{n^{2+\alpha}} < +\infty \iff 2+\alpha > 1 \iff \alpha > -1$

Diverge per $\alpha \leq -1$. □

Es. Det. l'insieme di convergenza di $\boxed{\sum_{n=3}^{+\infty} \frac{2n-1}{n^2-4}(e^{x-1})^n}$

$$\rightsquigarrow \sum_{n=3}^{+\infty} \underbrace{\frac{2n-1}{n^2-4}}_{a_n} t^n \qquad \sqrt[n]{a_n} \to 1 \qquad r_t = 1$$

$\Rightarrow$ converge $\forall\ |t| < 1$

per $t = -1$ converge per Leibniz

per $t = 1$ non converge perchè $\frac{2n-1}{n^2-4} \sim \frac{1}{n}$ per $n \to +\infty$

$\rightsquigarrow E_t = [-1, 1[$ la serie in t converge.

Ricord. $t = e^{x-1}$ $\Rightarrow -1 \leq e^{x-1} < 1 = e^0$ $\Rightarrow$ insieme di converg. della
$-1 \leq t < 1$
serie data $E =]-\infty, 1[$ ■

INTEGRALE IMPROPRIO

Es. Studiate al variare di $\alpha > 0$ la convergenza $\boxed{\displaystyle\int_0^{+\infty} \underbrace{\frac{\operatorname{arctg}\left(\frac{1}{\sqrt[3]{x}}\right)}{(x+1)^{3\alpha}\, x^{\alpha}}}_{f(x)}\, dx}$

per $x \to 0^+$ $\quad f(x) \sim \dfrac{\pi/2}{x^{\alpha}}$

per $x \to +\infty$ $\quad f(x) \sim \dfrac{\frac{1}{\sqrt[3]{x}}}{x^{3\alpha} \cdot x^{\alpha}} = \dfrac{1}{x^{\frac{1}{3}} \cdot x^{3\alpha+\alpha}} = \dfrac{1}{x^{\frac{1}{3}+4\alpha}}$

$\operatorname{arctg} \dfrac{1}{\sqrt[3]{x}} \sim \dfrac{1}{\sqrt[3]{x}}$ per $x \to +\infty$

Poiché $\displaystyle\int_0^1 \frac{1}{x^{\alpha}}\,dx < +\infty \iff \alpha < 1$

$\displaystyle\int_1^{+\infty} \frac{1}{x^{\frac{1}{3}+4\alpha}}\,dx < +\infty \iff \frac{1}{3} + 4\alpha > 1$

per il criterio del confronto asintotico abbiamo la converg. dell' integrale dato $\iff$ $\frac{1}{6} < \alpha < 1$. ■