Commenti alla lezione del 13/12/2005 (23ª lezione Corso)

<u>Riferimento bibliografico</u> : [2] ; Cap. 5 Sez. 5.3 (Teor. fond. calcolo integr. 5.2
Teor. di Torricelli 5.3)

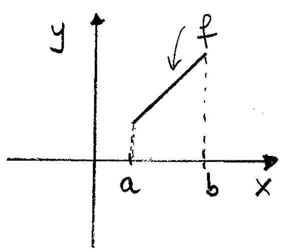
Def. pag. 219

Non tutte le funzioni ammettono primitiva; se però f è continua in $[a,b]$, allora essa possiede una primitiva (e quindi infinite).

<u>Def.</u> Data $f:[a,b]\to\mathbb{R}$ continua, definiamo la <u>funzione integrale</u> <u>(di f)</u>, $F:[a,b]\to\mathbb{R}$ mediante

$$F(x)=\int_a^x f(t)dt \qquad \text{per } x\in[a,b]$$

<u>OSS.</u> (a) Sia $f:[a,b]$ come in figura :



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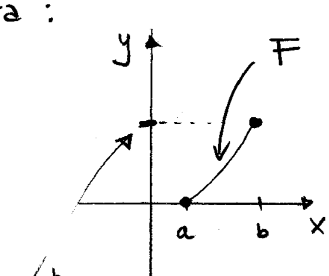
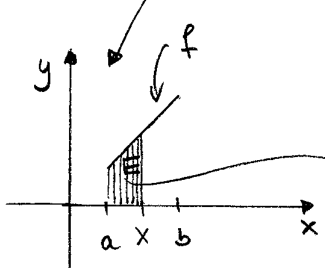


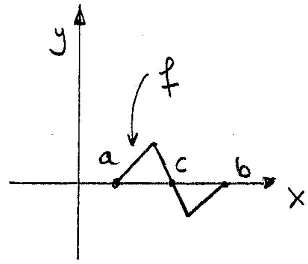
$$\int_a^b f(t)dt = \text{area}\left(\left\{(x,y)\in\mathbb{R}^2 : \begin{array}{l} a\le x\le b \\ 0\le y\le f(x)\end{array}\right\}\right)$$



$$\text{area}(E)=\int_a^x f(t)dt = F(x)$$

Se $x=a$, abbiamo $F(a)=\int_a^a f(t)dt=0$; man mano che x cresce verso b, cresce il valore di $F(x)$; il suo massimo lo raggiungerà se $x=b$. $\square$

(b) Sia $f:[a,b]$ come in figura :



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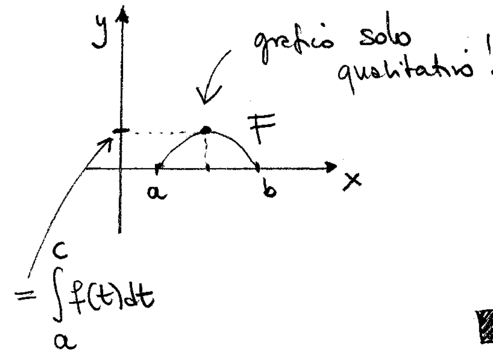


Abbiamo il seguente teorema fondamentale :

> <u>Teorema (teorema fondamentale del calcolo integrale)</u>:
> Sia $f:[a,b]\to\mathbb{R}$ continua. Allora la funzione integrale F è una primitiva di f; ossia F è derivabile in $[a,b]$ e
> $$F'(x) = f(x) \quad \forall\, x\in[a,b].$$

<u>NOTAZIONE</u> : Come conseguenza di questo teorema si usa la seguente notazione :

$$\boxed{\int f(x)dx = \text{integrale indefinito di } f \doteq \{\text{primitive di } f\}}$$

La conseguenza più importante del teorema sopra è il seguente

> <u>Teorema (di Torricelli)</u>: Sia $f:[a,b]\to\mathbb{R}$ continua. Sia G una primitiva di f in $[a,b]$, allora
> $$\int_a^b f(t)dt = G(b)-G(a) \quad \left(\Big[G(x)\Big]_a^b = G(x)\Big|_a^b \text{ notazione per } G(b)-G(a) \right)$$
> $$= \text{variazione di } G \text{ sull'intervallo } [a,b].$$

Dim. Poichè $G(x)$ e $F(x)=\int_a^x f(t)dt$ sono entrambe primitive di f su $[a,b]$, esse devono differire per una costante, cioè

$$\int_a^x f(t)dt = G(x)+c \qquad \forall x\in[a,b].$$

Per $x=a$ si ha $\quad 0=\int_a^a f(t)dt = G(a)+c \quad \Rightarrow c=-G(a).$

Per $x=b$ si ha $\quad \int_a^b f(t)dt = G(b)+c = G(b)-G(a).$ $\blacksquare$

Esercizio 1. Calcolate gli integrali definiti

i) $\int_0^1 (x^5+x^3)dx$; ii) $\int_0^4 \sqrt{x}\, dx$; iii) $\int_4^9 \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx$;

iv) $\int_{-2}^2 (x^2+3)^2 dx$; v) $\int_1^2 \left(e^x+\frac{1}{x}\right)dx$.

Svolgimento:

i) $$\int_0^1 (x^5+x^3)\,dx = \left[\frac{x^6}{6}+\frac{x^4}{4}\right]_0^1 = \frac{1}{6}+\frac{1}{4}=\frac{5}{12}.$$ $\square$

ii) $$\int_0^4 \sqrt{x}\,dx = \int_0^4 x^{\frac12}dx = \left[\frac{x^{\frac12+1}}{\frac32}\right]_0^4 = 2\cdot\frac{4^{3/2}}{3} = \frac{2\,(2)^3}{3} = \frac{16}{3}.$$ $\square$

iii) $$\int_4^9\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)dx = \int_4^9\left(x^{\frac12}-x^{-\frac12}\right)dx = \left[\frac{x^{3/2}}{\frac32}-\frac{x^{1/2}}{\frac12}\right]_4^9 =$$

$$=\left[\frac23\cdot 9^{3/2}-2\cdot 9^{\frac12}\right]-\left[\frac23\cdot 4^{3/2}-2\cdot 4^{1/2}\right]=$$

$$=\left[\frac{2}{\cancel{3}}\cdot 3^{\cancel{3}\,2}-2\cdot 3\right]-\left[\frac23\cdot 2^3-2\cdot 2\right]=(18-6)-\left(\frac{16}{3}-4\right)=$$

$$=\frac{32}{3}.$$ $\square$

iv) $$\int_{-2}^{2} (x^2+3)^2 dx = \int_{-2}^{2} (x^4 + 6x^2 + 9)\, dx = \left[\frac{x^5}{5} + 6\cdot\frac{x^3}{3} + 9x\right]_{-2}^{2} =$$

$$= \left[\frac{32}{5} + 2\cdot 8 + 18\right] - \left[-\frac{32}{5} - 2\cdot 8 - 18\right]$$

$$= 2\left(\frac{32 + 34\cdot 5}{5}\right) = \underline{\underline{80 + \frac{4}{5}}}\,.$$

$\square$

v) $$\int_{1}^{2} \left(e^x + \frac{1}{x}\right) dx = \left(e^x + \log|x|\right)_1^2 = \left(e^2 + \log 2\right) - \left(e + \cancel{\log} 1\right)$$

$$= \underline{\underline{e^2 - e + \log 2.}}$$

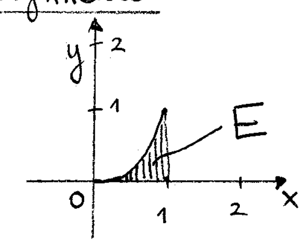
$\blacksquare$

Esercizio 2. Calcolate l'area della regione piana

i) $E = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le x^2\}$;

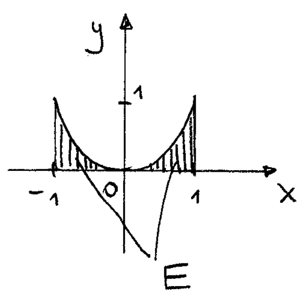
ii) $E = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 1,\ 0 \le y \le |x|^3\}$.

Svolgimento:

i) 

$$\text{area}(E) = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \underline{\underline{\frac{1}{3}}}\,.$$

$\square$

ii) 

$$\text{area}(E) = \int_{-1}^{1} |x|^3 dx = 2\int_0^1 x^3 dx = \left[\cancel{2}\cdot\frac{x^4}{\cancel{4}_{\,2}}\right]_0^1$$

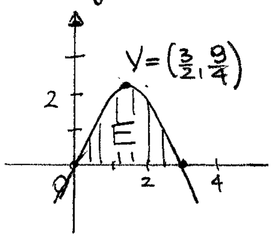
$$= \underline{\underline{\frac{1}{2}}}$$

$\blacksquare$

Esercizio 3. Determinate l'area della regione piana che sta sopra l'asse x e sotto il grafico della funzione $f(x) = 3x - x^2$.

Svolgimento: $f(x) = 3x - x^2 = x(3-x)$; nota anche che

$$f(x) = -\left(x - \tfrac{3}{2}\right)^2 + \tfrac{9}{4}$$

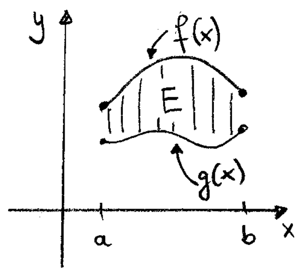


$$\text{area}(E) = \int_0^3 (3x - x^2)\,dx = \left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_0^3 =$$

$$= \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}$$

Oss.

ⓐ

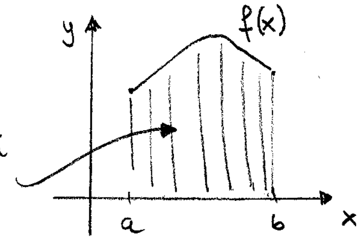


$$\text{area}(E) = \text{area}\left(\{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ 0 \le g(x) \le y \le f(x)\}\right)$$

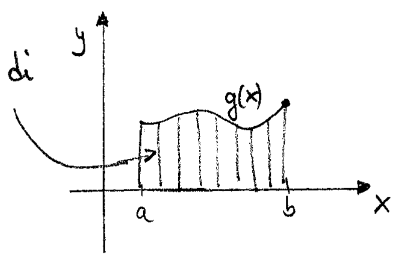
$$= \int_a^b (f(x) - g(x))\,dx .$$

Infatti

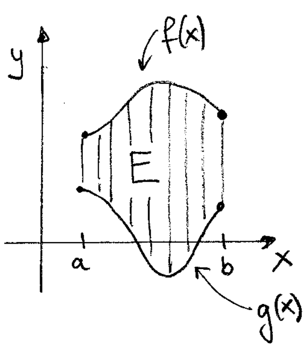
$\int_a^b f(x)\,dx =$ area di



$\int_a^b g(x)\,dx =$ area di



ⓑ

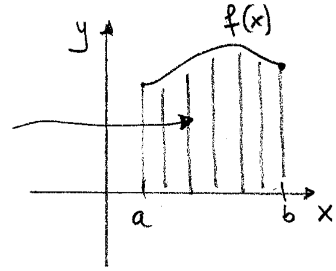


$$\text{area}(E) = \text{area}\left(\{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ g(x) \le y \le f(x)\}\right)$$

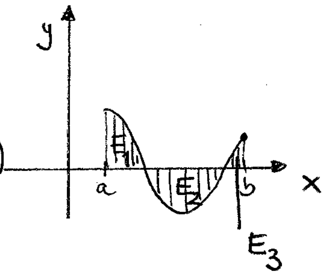
$$= \int_a^b (f(x) - g(x))\,dx .$$

Infatti

$$\int_a^b f(x)\,dx = \text{area di}$$



$$\int_a^b g(x)\,dx = \text{area}(E_1) - \text{area}(E_2) + \text{area}(E_3)$$

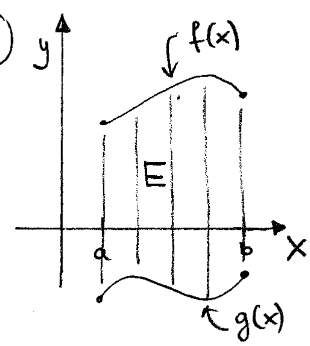


Quindi

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x)\,dx - \text{area}(E_1) + \text{area}(E_2) - \text{area}(E_3)$$

□

(c)



Anche qua si ottiene immediatamente che

$$\text{area}(E) = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b \left(f(x) - g(x)\right) dx$$

■

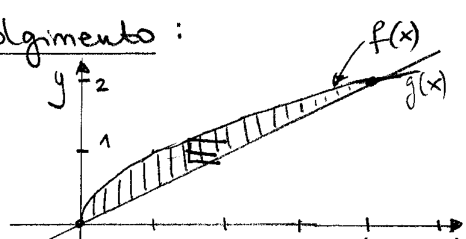
Esercizio 4. Determinate l'area della regione E specificata negli esercizi seguenti:

i) sotto $f(x) = \sqrt{x}$ e sopra $g(x) = \frac{x}{2}$;

ii) sopra $f(x) = |x|$ e sotto $g(x) = 12 - x^2$.

Svolgimento:

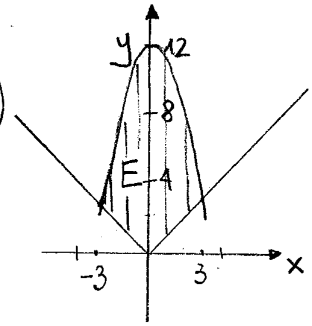
i)



$$\text{area}(E) = \int_0^4 \left(f(x) - g(x)\right) dx =$$
$$= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \frac{4}{3}.$$

□

ii)



$$\begin{cases} x \geq 0 \\ 12 - x^2 = x \end{cases} \iff \begin{cases} x \geq 0 \\ x^2 + x - 12 = 0 \end{cases} \Rightarrow x = 3$$

$$\text{area}(E) = 2\int_0^3 (12 - x^2 - x)\,dx =$$

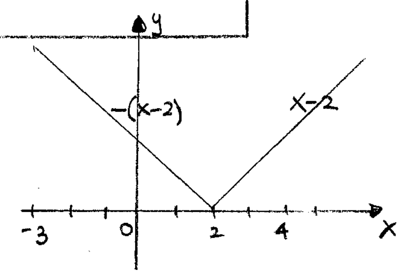
$$= 2\left[12x - \frac{x^3}{3} - \frac{x^2}{2}\right]_0^3 =$$

$$= 2\left[36 - 9 - \frac{9}{2}\right] = \underline{\underline{45.}}$$

Esercizio 5. Calcolate

i) $\displaystyle\int_{-3}^{5} |x-2|\,dx$; ii) $\displaystyle\int_1^2 \left(\frac{x^2+x+1}{\sqrt{x}}\right) dx$.

Svolgimento:



i) $\displaystyle\int_{-3}^{5} |x-2|\,dx =$

$$= \int_{-3}^{2} \left[-(x-2)\right] dx + \int_2^5 (x-2)\,dx =$$

$$= \left[-\frac{x^2}{2} + 2x\right]_{-3}^{2} + \left[\frac{x^2}{2} - 2x\right]_2^5 = \left[(-2+4) - \left(-\frac{9}{2} - 6\right)\right] +$$

$$+ \left[\left(\frac{25}{2} - 10\right) - (2-4)\right] = \underline{17}$$

ii) $$\int_1^2 \left(\frac{x^2+x+1}{\sqrt{x}}\right) dx = \int_1^2 \left(x^{3/2} + x^{1/2} + x^{-1/2}\right) dx =$$

$$= \frac{2}{5}\left[x^{5/2}\right]_1^2 + \frac{2}{3}\left[x^{3/2}\right]_1^2 + 2\left[x^{1/2}\right]_1^2$$

$$= \underline{\underline{\frac{74}{15}\sqrt{2} - \frac{46}{15}}}.$$