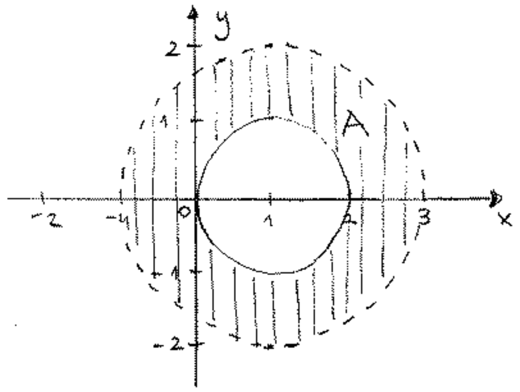
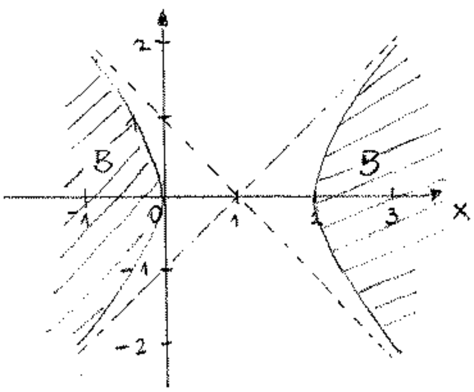
Università degli Studi di Trento – Facoltà di Scienze Cognitive – CdL in STPCA
Esame scritto di ANALISI MATEMATICA – Rovereto, 5 giugno 2008

FILA (A)

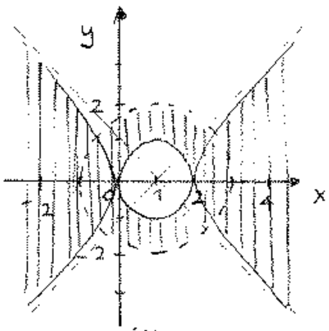
1) i) $A = \{(x,y)\in\mathbb{R}^2 : 0 \le x^2-2x+y^2 < 3\} = \{(x,y)\in\mathbb{R}^2 : 0 \le (x-1)^2+y^2-1 < 3\}$
$= \{(x,y)\in\mathbb{R}^2 : 1 \le (x-1)^2+y^2 < 4\}$;

$B = \{(x,y)\in\mathbb{R}^2 : -x^2+2x+y^2 \le 0\} = \{(x,y)\in\mathbb{R}^2 : x^2-2x-y^2 \ge 0\}$
$= \{(x,y)\in\mathbb{R}^2 : (x-1)^2-y^2 \ge 1\}$.

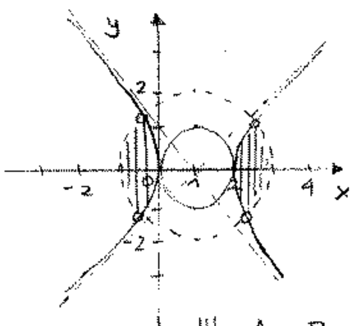




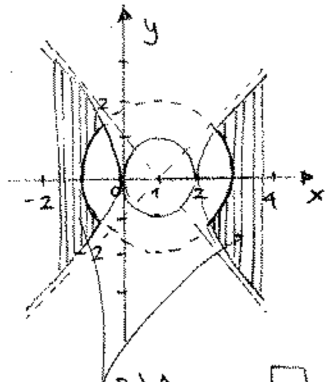
ii)



$\parallel$ $A\cup B$



$\parallel$ $A\cap B$

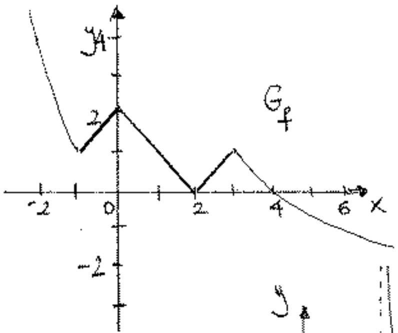


$B\setminus A$

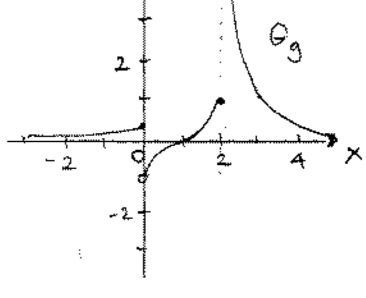
iii) $y=k \quad -2<k<2$

iv) $x=k \quad k\le 0 \quad$ o $\quad k \ge 2$

2) i) $f(x) = \begin{cases} x^2 & \text{se } x<-1 \\ ||x|-2| & \text{se } -1 \le x \le 3 \\ 1-\log_2(x-2) & \text{se } x>3 \end{cases}$



$g(x) = \begin{cases} (x-1)^3 & \text{se } 0<x\le 2 \\ \dfrac{1}{(x-2)^2} & \text{se } x\in\mathbb{R}\setminus\,]0,2] \end{cases}$



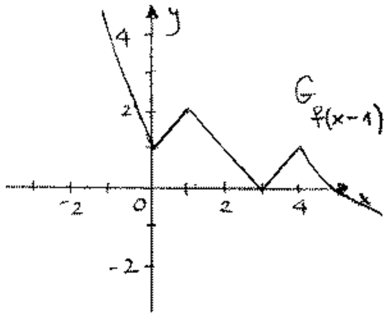
ii) f non è iniettiva (per esempio, $x_1 = 2 \neq x_2 = 4$, ma $f(x_1) = f(x_2) = 0$);
f è suriettiva, poiché $f(\mathbb{R}) = \mathbb{R}$.
g non è iniettiva (per esempio, $x_1 = 2 \neq x_2 = 3$, ma $g(x_1) = g(x_2) = 1$)
g non è suriettiva, poiché $g(\mathbb{R}) =]-1, +\infty[\subsetneq \mathbb{R}$. $\square$

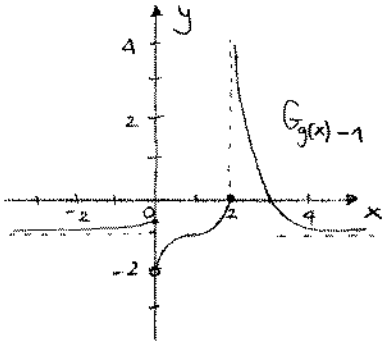
iii) f è continua, poiché negli intervalli $]-\infty, -1[$, $[-1, 3]$, $]3, +\infty[$ è continua; inoltre $\lim_{x \to -1^-} f(x) = 1 = f(-1)$
$\lim_{x \to 3^+} f(x) = 1 = f(3)$;

g è discontinua in $x = 0$: $g(0) = \frac{1}{4} \neq \lim_{x \to 0^+} g(x) = -1$
in $x = 2$: $g(2) = 1 \neq \lim_{x \to 2^+} g(x) = +\infty$.

In tutti gli altri punti di $\mathbb{R}$, g risulta continua. $\square$

iv)





3) i) $\log(2-|x|) - \log x^2 > 0 \iff \log(2-|x|) > \log x^2$

$\iff \begin{cases} 2-|x| > 0 \\ x^2 > 0 \\ 2-|x| > x^2 \end{cases} \iff \begin{cases} |x| < 2 \\ x \neq 0 \\ 2-|x| > x^2 \end{cases}$

$\iff \begin{cases} -2 < x < 0 \\ 2+x > x^2 \end{cases}$ o $\begin{cases} 0 < x < 2 \\ 2-x > x^2 \end{cases}$

$\iff \begin{cases} -2 < x < 0 \\ x^2 - x - 2 < 0 \end{cases}$ o $\begin{cases} 0 < x < 2 \\ x^2 + x - 2 < 0 \end{cases}$

ora $x^2 - x - 2 = 0 \iff x_{1/2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \begin{smallmatrix} \nearrow 2 \\ \searrow -1 \end{smallmatrix}$

$x^2 + x - 2 = 0 \iff x_{1/2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \begin{smallmatrix} \nearrow -2 \\ \searrow 1 \end{smallmatrix}$

quindi

$\begin{cases} -2 < x < 0 \\ -1 < x < 2 \end{cases}$ o $\begin{cases} 0 < x < 2 \\ -2 < x < 1 \end{cases} \Rightarrow S =]-1, 0[\cup]0, 1[$
$=]-1, 1[\setminus \{0\}$.

$\square$

$$e^{|x^2-1|} \leq e \iff |x^2-1| \leq 1 \iff -1 \leq x^2-1 \leq 1 \iff 0 \leq x^2 \leq 2$$

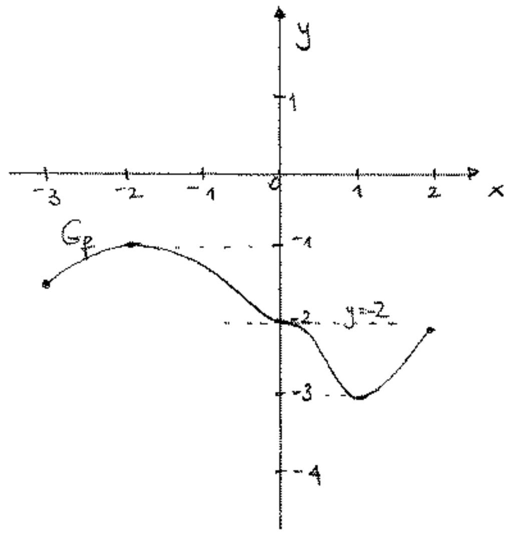
$$\Rightarrow \underline{S = [-\sqrt{2}, \sqrt{2}]}.$$

$$\frac{(\log_3 9)(\log_2 \frac{1}{4})}{\log x} \geq 0 \iff \frac{2 \cdot (-2)}{\log x} \geq 0 \iff \underline{x \in]0,1[}.$$

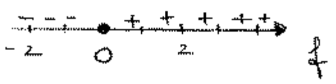
$$\frac{x}{|x+1|} < 0 \Rightarrow \underline{S =]-\infty, 0[\setminus \{-1\}}.$$

ii) $\displaystyle\sum_{n=1}^{4} \frac{(-1)^n}{n^n} = -1 + \frac{1}{4} - \frac{1}{27} + \frac{1}{16^2}$.

4) Esempio:



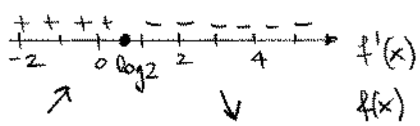
5) i) $f(x) = \dfrac{1-e^{-x}}{e^x} \quad \left(= \dfrac{e^x - 1}{e^{2x}} \right)$

- $\operatorname{dom} f = \mathbb{R}$; f è continua, essendo somme e rapporto di funz. continue; $\left(e^x > 0 \quad \forall x \in \mathbb{R}\right)$;
- 
- $\lim\limits_{x \to -\infty} f(x) = -\infty \qquad \lim\limits_{x \to +\infty} f(x) = 0$;
- $\operatorname{dom} f' = \mathbb{R}$ $\quad f'(x) = \dfrac{e^x \cdot e^{2x} - (e^x-1)\, e^{2x} \cdot 2}{e^{4x}} = \dfrac{e^{3x} - 2e^{3x} + 2e^{2x}}{e^{4x}}$

$$= \frac{e^{2x}(2 - e^x)}{e^{4x}} = \frac{2 - e^x}{e^{2x}}$$

$\Rightarrow x = \log 2$ pt. critico per f

$f(\log 2) = \underline{\frac{1}{4}}$



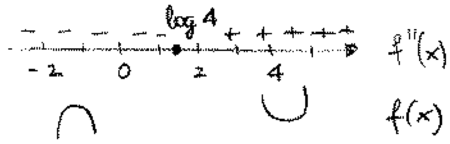
$x = \log 2$ è un pt. di massimo locale stretto per f.

- $\operatorname{dom} f'' = \mathbb{R}$ $\quad f''(x) = \dfrac{-e^x \cdot e^{2x} - (2-e^x)\, 2e^{2x}}{e^{4x}}$

$$= \frac{-e^x - 4 + 2e^x}{e^{2x}} = \frac{e^x - 4}{e^{2x}}$$

$x = \log 4$ pt. di flesso

$f(\log 4) = \dfrac{1 - \frac{1}{4}}{4} = \dfrac{3}{16}$



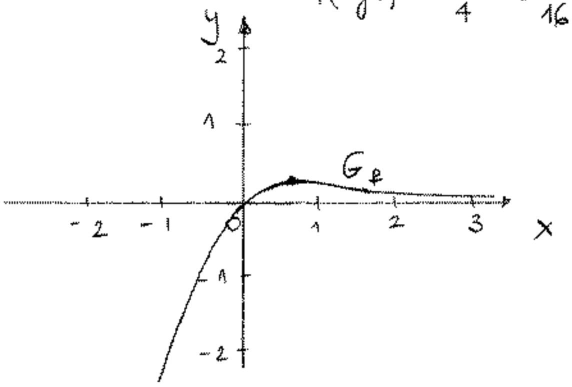


grafico qualitativo di f

$f(\log 4) = \frac{3}{16}$

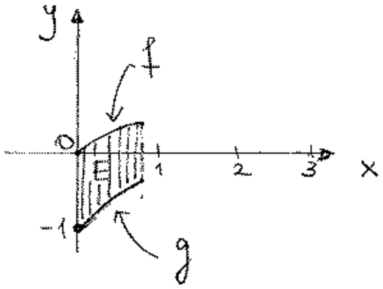
$f'(\log 4) = \dfrac{2-4}{16} = -\dfrac{2}{16} = -\dfrac{1}{8}$

ii) $y = f(\log 4) + f'(\log 4)(x - \log 4)$

$y = \frac{3}{16} - \frac{1}{8}(x - \log 4)$

$\underline{\underline{y = -\frac{1}{8}x + \frac{3}{16} + \frac{\log 4}{8}}}$ eq. retta tangente al grafico di f nel pt. $\left(\log 4, \frac{3}{16}\right)$.

iii)



$$\underline{\underline{\text{area}(E)}} = \int_0^{\log 2} (f(x) - g(x))\,dx = \int_0^{\log 2} \left(\frac{1-e^{-x}}{e^x} + \frac{1}{e^{2x}}\right) dx$$

$$= \int_0^{\log 2} \frac{e^x - \cancel{1} + \cancel{1}}{e^{2x}}\,dx = \int_0^{\log 2} \frac{1}{e^x}\,dx =$$

$$= \int_0^{\log 2} e^{-x}\,dx = \left[-e^{-x}\right]_0^{\log 2} = -\frac{1}{2} + 1 = \underline{\underline{\frac{1}{2}}}.$$

6) 1 2 3 4 5 6 7 8 9

una cifra di queste deve stare nel centro

$\underline{5 \cdot D_{8,4}}$