Esercitazione 11

Es 1 $\forall \beta \in \mathbb{R}$, sia S_β l'area del sottografico

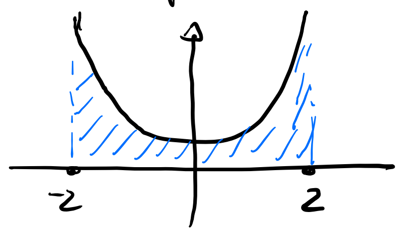
di $f(x) = x^2 + \beta$, $x \in [-2, 2]$

a) Disegnare la regione del piano.

Oss. $x^2 + \beta = 0 \iff x^2 = -\beta \iff x = \pm\sqrt{-\beta}$

$\Rightarrow$ Se $\beta \geq 0$, $\rightsquigarrow$

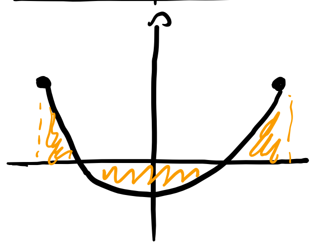
$$\text{Area}(S_\beta) = \int_{-2}^{2} f(x)\,dx$$



Cosa succede se $\beta < 0$?

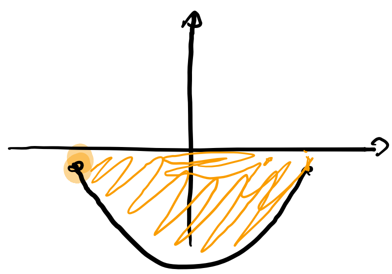
Oss che $\sqrt{-\beta} \leq 2 \iff -\beta \leq 4$. Quindi:

$-4 \leq \beta < 0$



$$S_\beta = \int_{-2}^{-\sqrt{-\beta}} f(x)\,dx + \int_{-\sqrt{-\beta}}^{\sqrt{-\beta}} -f(x)\,dx + \int_{\sqrt{-\beta}}^{2} f(x)\,dx$$

$\beta < -4$

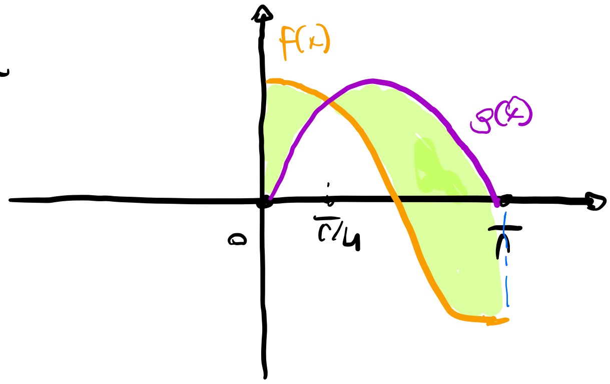


$$S_\beta = \int_{-2}^{2} -f(x)\,dx$$

Es 2 Calcola l'area compresa tra i grafici di

$f(x) = \cos(x)$, $g(x) = \sin(x)$ e le rette

$x = 0$ e $x = \pi$.



Oss che, su $[0, \pi]$,

$f(x) = g(x) \iff x = \frac{\pi}{4}$.

Inoltre $f(x) \geq g(x)$ su $[0, \pi/4]$ e $f(x) \leq g(x)$ su $[\pi/4, \pi]$.

Allora vale che

$$\text{Area} = \int_0^{\pi/4} f - g + \int_{\pi/4}^{\pi} g - f$$
$$= \int_0^{\pi/4} f - g - \int_{\pi/4}^{\pi} f - g.$$

Oss che $\int \cos x - \sin x = \sin x + \cos x + c \overset{TB}{\Longrightarrow}$

$$\text{Area} = \left[\cos x + \sin x\right]_0^{\pi/4} - \left[\cos x + \sin x\right]_{\pi/4}^{\pi}$$
$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 - \left(-1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

Es 3 Calcolare i seguenti integrali definiti.

a) $\displaystyle\int_1^2 \frac{x + 2\sqrt[3]{x}}{x^2}\,dx$

Sostituzione: $\sqrt[3]{x} = s \Rightarrow x = s^3, \quad dx = 3s^2 ds$

$x_0 = 1, \ x_1 = 2 \Rightarrow s_0 = \sqrt[3]{x_0} = 1, \quad s_1 = \sqrt[3]{x_1} = \sqrt[3]{2}$

$$\int_1^2 \frac{x + 2\sqrt[3]{x}}{x^2}\,dx = \int_1^{\sqrt[3]{2}} \frac{s^3 + 2s}{s^6} \cdot 3s^2\,ds$$

$$= \int_1^{\sqrt[3]{2}} \frac{3}{s} + \frac{6}{s^3}\,ds = \left[3\log|s| - \frac{3}{s^2}\right]_1^{\sqrt[3]{2}}$$

$$= 3\log\sqrt[3]{2} - \frac{3}{\sqrt[3]{4}} + 3 = \log 2 - \frac{3}{\sqrt[3]{4}} + 3$$

In alternativa

$$\int_1^2 \frac{x + 2\sqrt[3]{x}}{x^2}\,dx = \int_1^2 \frac{1}{x} + 2x^{-\frac{5}{3}}$$

$$= \left[\log|x| - \cancel{2} \cdot \frac{3}{\cancel{2}} x^{-\frac{2}{3}}\right]_1^2$$

$$= \log 2 - 3\frac{1}{\sqrt[3]{4}} + 3$$

MORALE: PRIMA DI SOSTITUIRE, CERCATE SE CI SONO SOLUZIONI PIÙ IMMEDIATE!

b) $\displaystyle\int_9^{16} \frac{\sqrt{x}-3}{x-3\sqrt{x}+2}$

Sostituzione $\sqrt{x}=s$, $x=s^2$, $dx = 2s\,ds$

$$\int_9^{16} \frac{\sqrt{x}-3}{x-3\sqrt{x}+2}\,dx = \int_3^4 \frac{s-3}{s^2-3s+2}\cdot 2s\,ds$$

$$=2\int_3^4 \frac{s^2-3s}{s^2-3s+2}\,ds = 2\int_3^4 \frac{s^2-3s+2}{s^2-3s+2}\,ds - 4\int_3^4 \frac{1}{s^2-3s+2}$$

$$2 - 4\int_3^4 \frac{1}{s^2-3s+2}$$

Oss. $s^2-3s+2 = (s-1)(s-2)$

Cerchiamo $A, B \in \mathbb{R}$ t.c. $\dfrac{A}{s-1} + \dfrac{B}{s-2} = \dfrac{1}{s^2-3s+2}$

$$\frac{A}{s-1} + \frac{B}{s-2} = \frac{As-2A+Bs-B}{(s-1)(s-2)}$$

$$\Longleftrightarrow \begin{cases} A=-B \\ -2A-B=1 \end{cases} \Longleftrightarrow \begin{cases} A=-B \\ 2B-B=1 \end{cases} \Longleftrightarrow \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$-4\int_3^4 \frac{1}{s^2-3s+2}\,ds = 4\int_3^4 \frac{1}{s-1}\,ds - 4\int_3^4 \frac{1}{s-2}\,ds$$

$$= 4\left[\log|s-4|\right]_3^4 - 4\left[\log|s-2|\right]_3^4 = 4\log 3 - 4\log 2 - 4\log 2$$

$$\Rightarrow \int = 2 + 4\log\frac{3}{4}$$

c) $\int_0^{e-\frac{1}{e}} \sqrt{4+x^2}\, dx$

Sostituzione $x = 2\sinh(s)$, $dx = 2\cosh(s)\, ds$

$$\int \sqrt{4+x^2}\, dx = \int \sqrt{4+4\sinh^2 s}\, 2\cosh s\, ds$$

$$= \int \sqrt{4\cosh^2 s}\, \cosh(s)\, ds = 4\int \cosh^2 s\, ds$$

IPP $= 4\cosh s \sinh s - 4\int \sinh^2 s$

$$= 4\cosh s \sinh s - 4\int (\cosh^2 s - 1)$$

$$= 4\cosh s \sinh s + 4\int 1 - 4\int \cosh^2 s\, ds$$

$$\Rightarrow \int \cosh s = \frac{1}{2}\cosh s \sinh s + \frac{s}{2}$$

$\Rightarrow$

$$\int \sqrt{4+x^2} = 2\cosh s \sinh s + 2s$$

Oss. $x = 2\sinh(s) = e^s - e^{-s}$.

Quindi $x = 0 \Leftrightarrow s = 0$ e $x = e - \frac{1}{e} \Leftrightarrow s = 1$

$$\int_0^{e-\frac{1}{e}} \sqrt{4+x^2} = 2\left[\cosh s \sinh s + s\right]_0^1 .$$

d) $\int \frac{1}{\sin x + 1} \, dx$

Sostituzione $\quad t = \tan\left(\frac{x}{2}\right) \Rightarrow x = 2\arctan(t)$

$dx = \frac{2}{1+t^2} \, dt$. Oss che $\frac{2t}{1+t^2} = \frac{2\,\frac{\sin(\frac{x}{2})}{\cos(x/2)}}{1+\frac{\sin^2(\frac{x}{2})}{\cos^2(x/2)}}$

$$= \frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)} = \sin x$$

(NB Formule simili esistono per $\cos(x)$)

$$\Rightarrow \int \frac{1}{\sin x + 1} \, dx = 2\int \frac{1}{1+t^2} \cdot \frac{1}{\frac{2t}{1+t^2}+1}$$

$$= 2\int \frac{1}{\cancel{1+t^2}} \cdot \frac{\cancel{1+t^2}}{1+2t+t^2} \, dt = \int \frac{2}{(1+t)^2}$$

$$= -\frac{2}{(1+t)} + c = -\frac{2}{1+\tan(\frac{x}{2})} + c$$

Es 4 Convergenza degli integrali impropri

$$\int_0^1 \underbrace{\frac{\sin x}{\sqrt{x}\,\sqrt[4]{1-x}}}_{=f(x)}$$

<u>In 0.</u> Oss che $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x + o(x)}{\sqrt{x}} = 0.$

Quindi f è estendibile con continuità in 0

$\Rightarrow$ Non è un int. improprio in 0.

<u>In 1</u> $f(x) \sim \frac{\sin 1}{\sqrt[4]{1-x}}$ se $x \to 1^-$.

Dunque, confrontando $\frac{\sin 1}{\sqrt[4]{1-x}}$ con $\frac{1}{\sqrt[4]{1-x}}$,

ottengo che c'è convergenza in 1.

$\Rightarrow \int_0^1 f(x)\,dx$ è convergente.

Es 5 Determina $\alpha, \beta \in \mathbb{R}$ tc

$$\int_0^{+\infty} \frac{1}{x^\alpha (4+9x)^{\beta+1}} dx \quad \text{converge}$$

In 0 : oss che $f(x) \sim \frac{1}{4x^\alpha}$.

Per il C.A, conv in 0 $\iff$ $\boxed{\alpha < 1}$

A $+\infty$: $f(x) \sim \frac{1}{9^{\beta+1} x^{\alpha+\beta+1}}$

Per CA, conv $\iff$ $\alpha+\beta+\cancel{1} > \cancel{1}$

~~α, β~~ devono quindi soddisfare $\begin{cases} \alpha < 1 \\ \alpha > -\beta \end{cases}$

ovvero $\boxed{-\beta < \alpha < 1}$

Calcola $\int_0^{+\infty} \frac{1}{\sqrt{x}} \cdot \frac{1}{(4+9x)} dx$.

Poiché l'integrale converge, abbiamo che

$$\int_0^{+\infty} f(x) = \int_0^1 f(x) + \int_1^{+\infty} f(x) =$$

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}} \cdot \frac{1}{(4+9x)} + \lim_{M \to +\infty} \int_0^M \frac{1}{\sqrt{x}(4+9x)}$$

Primitiva di f?

Sostituzione $\sqrt{x} = s \quad x = s^2 \quad dx = 2s\,ds$

$$\int \frac{1}{\sqrt{x}\,(4+9x)}\,dx = \int \frac{1}{s(4+9s^2)}\,2s\,ds$$

$$= \int \frac{2}{4+9s^2} = \frac{1}{2}\int \frac{1}{1+\left(\frac{3}{2}s\right)^2}\,ds$$

Sostituzione $\frac{3}{2}s = t \quad s = \frac{2}{3}t \quad ds = \frac{2}{3}dt$

$$= \frac{1}{3}\int \frac{1}{1+t^2} = \frac{1}{3}\arctan(t) + c$$

$$= \frac{1}{3}\arctan\left(\frac{3}{2}s\right) + c = \frac{1}{3}\arctan\left(\frac{3}{2}\sqrt{x}\right) + c$$

$\Rightarrow$

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} f(x) \overset{\lim_{\varepsilon\to 0}}{=} \left[\frac{1}{3}\arctan\left(\frac{3}{2}\sqrt{x}\right)\right]_{\varepsilon}^{1} = \frac{1}{3}\arctan\left(\frac{3}{2}\right)$$

$$\lim_{M \to +\infty} \int_{1}^{M} f(x) = \lim_{M\to+\infty} \left[\frac{1}{3}\arctan\left(\frac{3}{2}\sqrt{x}\right)\right]_{1}^{M} = \frac{\pi}{6} - \frac{1}{3}\arctan\left(\frac{3}{2}\right)$$

$$= \int_{0}^{+\infty} f(x)\,dx = \frac{\pi}{6}$$

Es 6 Convergenza, al variare di $\alpha \in \mathbb{R}$, di

$$\int_1^2 \frac{1-\cos(x-1)}{(x^2-1)^\alpha (2-x)^{3-\alpha}}$$

Oss.

$$\int_1^2 \frac{1-\cos(x-1)}{(x^2-1)^\alpha (2-x)^{3-\alpha}} = \int_1^2 \frac{1-\cos(x-1)}{(x-1)^\alpha (x+1)^\alpha (2-x)^{3-\alpha}}$$

Sostituzione $x-1=s$, $x=s+1$, $dx=ds$

$$= \int_0^1 \underbrace{\frac{1-\cos s}{(s+2)^\alpha s^\alpha (1-s)^{3-\alpha}}}_{=f(s)}$$

In 0. NB $1-\cos s = \cancel{1}-\cancel{1}+\frac{s^2}{2}+o(s^2)$

$$\Rightarrow f(s) = \frac{s^2+o(s^2)}{2(s+2)^\alpha s^\alpha (1-s)^{3-\alpha}}$$

$$= \frac{1+\frac{o(s^2)}{s^2}}{2(s+2)^\alpha s^{\alpha-2}(1-s)^{3-\alpha}} \sim \frac{1}{2^{\alpha+1}} \cdot \frac{1}{s^{\alpha-2}}$$

Quindi converge $\iff \alpha-2<1 \iff \boxed{\alpha<3}$

In 1 $\quad f(s) \sim \frac{1-\cos(1)}{3^\alpha} \cdot \frac{1}{(1-s)^{3-\alpha}}$

Quindi converge $\iff 3-\alpha<1 \iff \boxed{\alpha>2}$

$\Rightarrow$ converge $\iff \boxed{2<\alpha<3}$