Esercizio 1

Determinare sup, inf, max e min dei seguenti insiemi:

i) $A=\left\{ \frac{n}{n+1} : n\in\mathbb{N} \right\}$

ii) $B=\left\{ 2(-1)^n - \frac{1}{2n} : n\in\mathbb{N}_{\neq 0} \right\}$

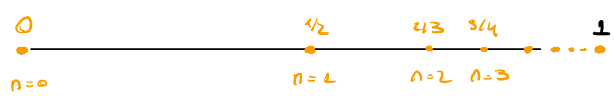
iii) $C=\left\{ \frac{n}{m} : n,m\in\mathbb{N}_{\neq 0},\ 0<m<n \right\}$

Soluzione

i) Oss. che $\frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1}$

Quindi $A=\left\{ 1-\frac{1}{n+1} : n\in\mathbb{N} \right\}$.

A:



- Poiché $0\in A$ e $\frac{n}{n+1}\geq 0\ \forall n$, allora $\inf A = \min A = 0$.
- Dimostriamo che $1=\sup A$. NB $1\notin A \Rightarrow A$ non ha massimo.

Proviamo che $\forall \varepsilon>0\ \exists n_\varepsilon\in\mathbb{N}$ t.c. $1-\frac{1}{n+1}\geq 1-\varepsilon \iff$
$\frac{1}{n+1}\leq\varepsilon$, ovvero $\iff 1\leq n\varepsilon+\varepsilon \iff n\geq\frac{1-\varepsilon}{\varepsilon}$.

ii) $B = \overbrace{\left\{2 - \frac{1}{2n}, n \text{ pari}, n \geq 2\right\}}^{B'} \cup \overbrace{\left\{-2 - \frac{1}{2n}, n \text{ dispari}\right\}}^{B''}$

Oss che $\forall x \in B', \forall y \in B'', \ y \leq x$.

$\Rightarrow$ Per il sup guardiamo B', per l'inf B''.

Proviamo che $\sup B = \sup B' = 2$ (<u>come sopra</u>)

Infine, $\forall n$ dispari, $-2 - \frac{1}{1 \cdot 2} \leq -2 - \frac{1}{2n}$.

$\Rightarrow \inf B = \min B = -\frac{5}{2}$.

iii) Da $m < n$ segue che $\frac{n}{m} > 1$. $\Rightarrow$ 1 è <u>minorante</u>

Inoltre $\left\{\frac{n+1}{n}, n > 0\right\} \subseteq C \Rightarrow \inf C = 1$

Infine $\{n \in \mathbb{N}, n \geq 2\} \subseteq C \Rightarrow \sup C = +\infty$.

Esercizio 2

Risolvere in $\mathbb{C}$ le seguenti equazioni:

i) $\frac{1}{|z|\,\bar{z}} = e^{i\frac{\pi}{4}}$

ii) $\bar{z}\,(\mathrm{Im}(z) - \mathrm{Re}(z)) = z$

iii) $z^3 = 1$

iv) $2z^2 + \bar{z} = -1$

i) Oss che $\left| e^{i\frac{\pi}{4}} \right| = 1$, da cui

$$\left| \frac{1}{|z|\,\bar{z}} \right| = 1 \overset{|\bar{z}|=|z|}{\Longrightarrow} \frac{1}{|z|^2} = 1 \Longrightarrow |z| = 1$$

Quindi $\frac{1}{\bar{z}} = e^{i\pi/4}$. Ricordando che $z \cdot \bar{z} = |z|^2 = 1$,

otteniamo $z = e^{i\pi/4}$

ii) Sia $z = a+ib$. L'eq. diventa

$$(a-ib)(b-a) = a+ib \text{, ovvero}$$

$$a(b-a) + i(a-b)b = a+ib \text{, da cui}$$

$$\begin{cases} a(b-a) = a \\ b(a-b) = b \end{cases}$$

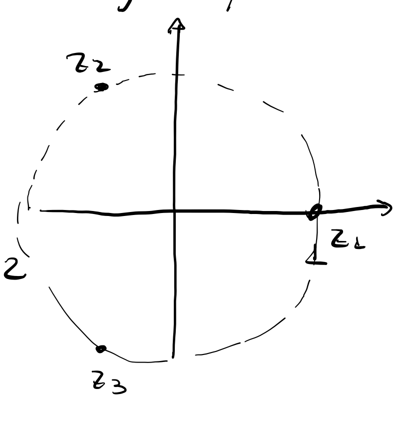
<u>Se $a=0$</u> allora ci rimane da imporre $-b^2 = b$, ovvero $b(b+1) = 0$, ovvero $\underline{b=0}$ o $\underline{b=-1}$.
Troviamo quindi $z=0$ o $z=-i$.

<u>Se $a \neq 0$</u>, allora dalla prima eq $\Rightarrow$ $b = a+1$.
Sostituendo nella seconda ottengo $-b = b$, ovvero $b=0$.
Quindi $a = -1 \Rightarrow \underline{z = -1}$

iii) $z^3 = 1$. Radici terze di 1. Messo $w = \rho e^{i\theta}$, allora

$$\sqrt[k]{w} = \sqrt[k]{\rho}\, e^{i\frac{\theta + j2\pi}{k}}, \quad j = 0 \dots k-1$$

Allora, poiché $1 = e^{i\cdot 0}$, $z_j = e^{i\frac{j}{3}2\pi}$, $j = 0,1,2$



iv) $2z^2+\bar{z}=-1$. Sia $z=a+ib$

$\Rightarrow z^2=(a+ib)(a+ib)=a^2+2aib-b^2$

$\Rightarrow 2a^2-2b^2+4aib+a-ib=-1$

$$\Rightarrow \begin{cases} 2a^2-2b^2+a=-1 \\ (4a-1)\,b=0 \end{cases}$$

Se $b=0$, $2a^2+a+1=0$, $a=\frac{-1\pm\sqrt{1-8}}{\;}$ ↯

$\Rightarrow$ No soluz se $b=0$

Se $b\neq 0$, allora $4a=1$, da cui $a=\frac{1}{4}$.

$\Rightarrow \frac{1}{8}-2b^2+\frac{1}{4}=-1 \Rightarrow -2b^2=-\frac{11}{8}$, $b=\pm\sqrt{\frac{11}{16}}$

Esercizio 3

Siano $f, g, h : \mathbb{C} \to \mathbb{C}$ definite da

$$f(z) = z + z_0, \quad g(z) = iz, \quad h(z) = z^2$$

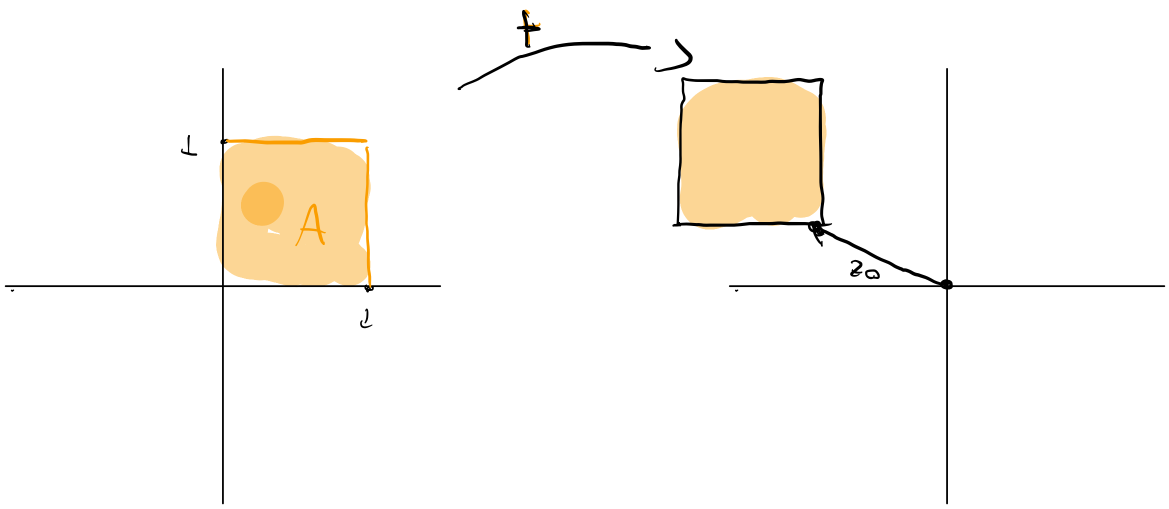
Siano $A, B, C \subseteq \mathbb{C}$ t.c.

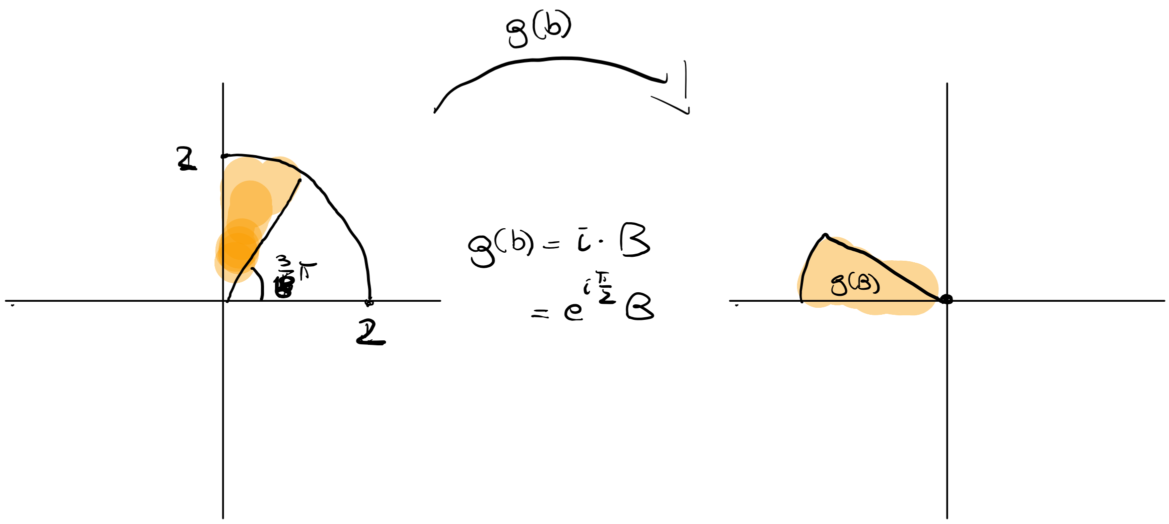
$$A = \{ z : \operatorname{Re}(z), \operatorname{Im}(z) \in [0, 1] \}$$

$$B = \{ z : |z| \leq 2, \ \arg(z) \in (\tfrac{3}{8}\pi, \tfrac{7}{12}\pi) \}$$

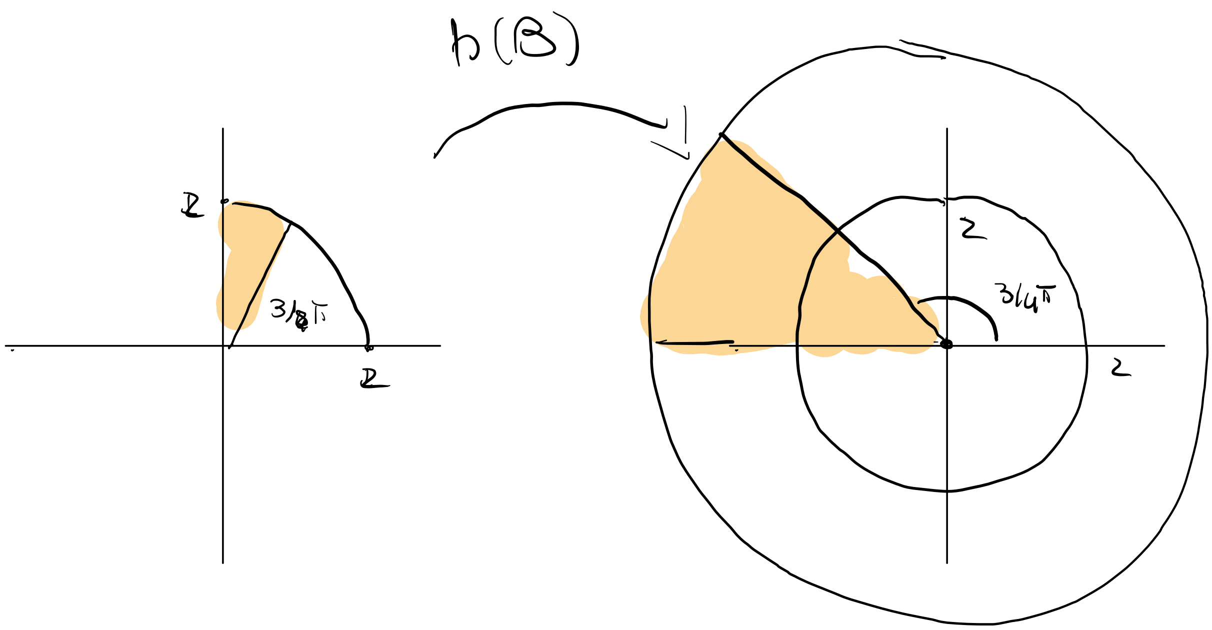
$$C = \{ z : 1 \leq |z| \leq 2 \}$$

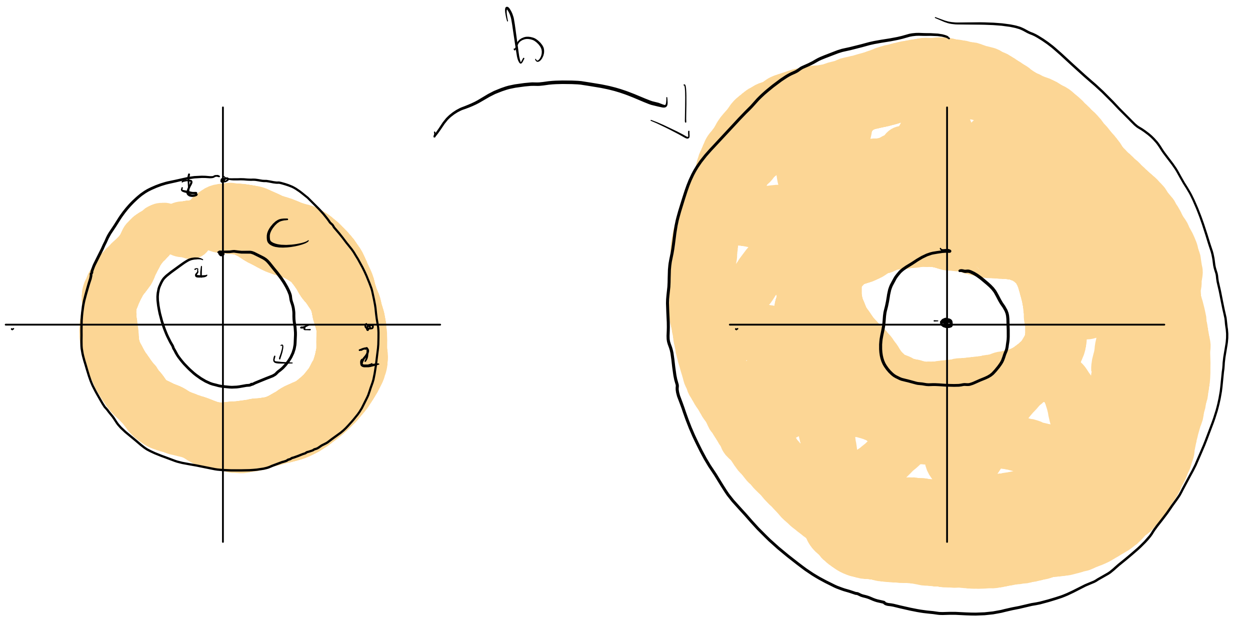
Determina $f(A)$, $g(B)$, $h(A)$, $h(C)$ e descrivi l'azione di queste funzioni.





$$g(b) = i \cdot B = e^{i\frac{\pi}{2}} B$$





h
C
1
2

Esercizio 4

Rappresenta in $\mathbb{C}$ i seguenti insiemi.

$$A = \left\{ z : \frac{|z-1|}{|z+1|} = 1 \right\}$$

$$B = \left\{ z : |z-i-2| < |z+2| ,\ |z| \geq 1 \right\}$$

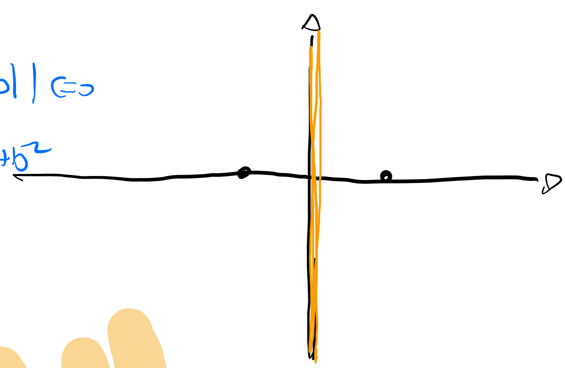
A: $|z-1| = |z+1| \iff |z-1| = |z-(-1)|$

$z = a+ib \Rightarrow$

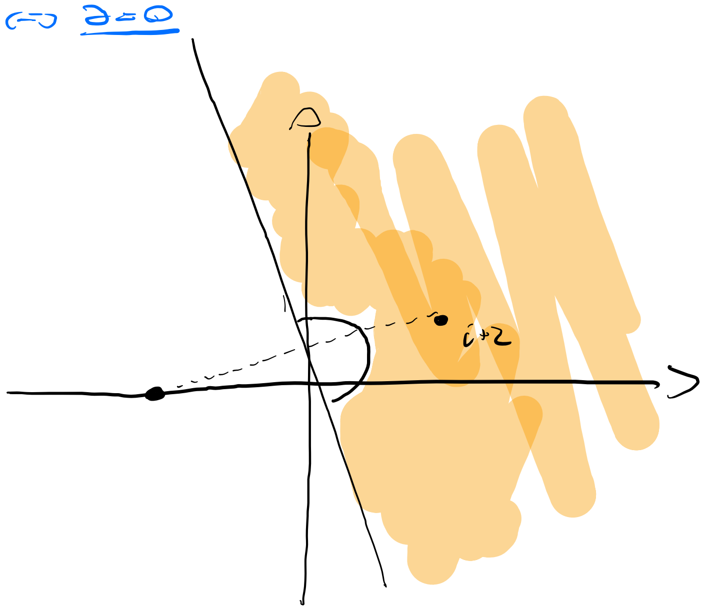
$|(a-1)+ib| = |(a+1)+ib| \iff$

$a^2+2a+1+b^2 = a^2-2a+1+b^2$

$\iff \underline{a=0}$



B



Esercizio 5

Dimostra le seguenti disuguaglianze:

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \overset{a)}{\leq} \sqrt{ab} \overset{b)}{\leq} \frac{a+b}{2} \overset{c)}{\leq} \sqrt{\frac{a^2+b^2}{2}}$$

a) $$\frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2}{\frac{a+b}{ab}} = \frac{2ab}{a+b}$$

$$\Rightarrow \frac{2ab}{a+b} \leq \sqrt{ab} \iff \frac{2\sqrt{ab}}{a+b} \leq 1$$

$$\iff \sqrt{ab} \overset{b)}{\leq} \frac{a+b}{2} \iff$$

$$4ab \leq a^2+2ab+b^2 \iff a^2-2ab+b^2 \geq 0 \quad \underline{\text{OK}}$$

c) $$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \iff \frac{a^2}{2}+ab+\frac{b^2}{2} \leq a^2+b^2 \iff \frac{a^2}{2}-ab+\frac{b^2}{2} \geq 0 \quad \underline{\text{OK}}$$